Decidability of a Description Logic over Infinite-Valued Product Logic

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Abstract

This paper proves that validity and satisfiability of assertions in the Fuzzy Description Logic based on infinite-valued Product Logic with universal and existential quantifiers (which are non-interdefinable) is decidable when we only consider quasi-witnessed interpretations. We prove that this restriction is neither necessary for the validity problem (i.e., the validity of assertions in the Fuzzy Description Logic based on infinite-valued Product Logic is decidable) nor for the positive satisfiability problem, because quasi-witnessed interpretations are particularly adequate for the infinite-valued Product Logic. We give an algorithm that reduces the problem of validity (and satisfiability) of assertions in our Fuzzy Description Logic (restricted to quasi-witnessed interpretations) to a semantic consequence problem, with finite number of hypothesis, on infinite-valued propositional Product Logic.

1. Introduction

Obtaining methods and tools suited to give a high-level description of the world and to implement intelligent systems plays a key role in the area of Knowledge Representation and Reasoning systems. Description Logics (DLs) are knowledge representation languages (particularly suited to specify formal ontologies), which have been studied extensively over the last two decades; a comprehensive reference manual of the field is (Baader et al. 2003). But in real applications the knowledge used is usually imperfect and has to address situations of uncertainty and vagueness. From a real world viewpoint, it is easy to find domains where concepts like "patient with a high fever" and "person living near a pollution source" have to be considered. The *vagueness* aspect has suggested to model DL concepts as fuzzy sets. Fuzzy Sets and Fuzzy Logic were born to deal with the problem of approximate reasoning (Zadeh 1965; 1975). Their first developments were characterized by the applications that gave rise to various semantic approaches to this problem.

In recent times, formal logic systems have been developed for such semantics, and the logics based on triangular norms (t-norms) have become the central paradigm of fuzzy logic. The development of this field of research is intimately linked to the book "Metamathematics of Fuzzy Logics" (Hájek 1998), which shows the connection of fuzzy logic systems with many-valued residuated lattices based on continuous t-norms. There are three basic continuous t-norms: minimum, Łukasiewicz and product t-norm; basic is used in the sense that all continuous t-norms can be obtained from these three ones using the "ordinal sum" construction. For each one of these three t-norms a propositional and a first order logical system have been studied in the literature.

We point out that in this paper we deal with logics given by the standard semantics in the fuzzy tradition, and not by the general semantics (cf. (Hájek 1998)). The language of these logics takes as primitive connectives the multiplicative conjunction $\odot$, its residuum implication $\rightarrow$ and the falsum constant $\perp$; while the intended semantics of $\odot$ is the corresponding t-norm $*$, the semantics of $\rightarrow$ is given by the residuum of the t-norm

$$x \Rightarrow y := \max\{z \in [0,1] : x * z \leq y\},$$

and $\perp$ is interpreted as 0. It is well known that simply using these three connectives we can define a new constant $\top$ as well as new connectives $\wedge$ and $\vee$ whose intended semantics are 1 and the lattice operations over $[0,1]$ with its natural order; moreover, it is common to introduce a negation $\neg$ defined by $\neg\varphi := \varphi \rightarrow \perp$, and the biconditional $\leftrightarrow$ defined by $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \odot (\psi \rightarrow \varphi)$. Complete (for finite theories) Hilbert style axiomatizations for the propositional logics defined by the three basic continuous t-norms can be found in (Hájek 1998); it is also proved there that the problem of a formula being valid in these propositional logics is, in the three cases, NP-complete. On the other hand, the behaviour of the first order logics[1] introduced by these three t-norms is not so nice: while in the minimum case a recursively axiomatizable logic is obtained, this is not true for the other two: Łukasiewicz t-norm introduces a Π_2-complete logic and product t-norm is even worst introducing a non arithmetical one (Montagna 2001).

Since classical DL $\mathcal{ALC}$ can be seen as a fragment of first order classical logic, Hájek proposed in (Hájek 2005) to introduce the fuzzy version, one for each t-norm, of this DL as a fragment of its first order fuzzy logic. In this paper we will use the notation $*-\mathcal{ALE}$ to denote the fuzzy DL defined by

[1]The intended semantics of the quantifiers $\forall$ and $\exists$ corresponds to the infimum and the supremum (of a subset of $[0,1]$), and hence it is a generalization of the semantics in the classical case.

Hájek using the t-norm $*$. In the concepts of these fuzzy DLs we allow to use the universal and existential (commonly denoted by $\mathcal{E}$) quantifier constructors, together with constructors for the three primitive connectives used in propositional fuzzy logics. We have said above that it is common to introduce a negation $\neg$, but it is worth noticing that in general this negation is not involutive; for this reason we prefer to use the name $*-\mathcal{ALE}$ for these logics rather than the $*-\mathcal{ALC}$ used in (Hájek 2005). In our opinion, the name $*-\mathcal{ALC}$ is the appropriate for the case that $*-\mathcal{ALE}$ is expanded with a constructor $\sim$ whose semantics corresponds to the involutive negation given by $\sim x := 1 - x$ (this is the approach used in (García-Cerdaña, Armengol, and Esteva 2010)).

In his paper, Hájek defines a concept C to be 1-*satisfiable in* $*-\mathcal{ALE}$ in case that there is some interpretation and some object a in the domain of the interpretation such that the truth value associated with $C(a)$ is 1. This definition can be generalized in the obvious way to r-*satisfiability* (for every $r \in [0, 1]$). We stress that we are not considering the problem whether a given concept is 1-satisfiable in a interpretation in the sense of being 1-satisfiable in all objects of the domain of the interpretation.[2] Analogously to the satisfiability case, Hájek also defines a concept C to be *valid in* $*-\mathcal{ALE}$ if for every interpretation and every object a in the domain of the interpretation the truth value of $C(a)$ is 1. We will use the notation Sat^*_r and Val^* to denote the set of concepts that are, respectively, r-satisfiable and valid in $*-\mathcal{ALE}$.

The main result in (Hájek 2005) says that if $*$ is Łukasiewicz t-norm, then the sets Sat^*_1 and Val^* are decidable; an easy consequence of this fact is that for every $r \in \mathbb{Q} \cap [0, 1]$, Sat^*_r is also decidable. The proof consists on two claims. The first claim is a general one: for every t-norm, the problems of 1-satisfiability and validity in finite interpretations are decidable problems. This is proved using a reduction[3] of the problem to the propositional fuzzy logic given by the corresponding t-norm; the idea behind this reduction is the fact that finite interpretations can be codified using a finite number of propositional formulas. The second claim is a particular one of Łukasiewicz: for every $r \in [0, 1]$, a concept is r-satisfiable iff it is r satisfiable in some finite interpretation. The proof of this fact is based on the notion of witnessed interpretation: an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is *witnessed* in case that

(wit∃) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\exists R.C)^{\mathcal{I}}(a) \;=\; R^{\mathcal{I}}(a, b) * C^{\mathcal{I}}(b),$$

(wit∀) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\forall R.C)^{\mathcal{I}}(a) \;=\; R^{\mathcal{I}}(a, b) \Rightarrow C^{\mathcal{I}}(b).$$

[2] Thus, the problem we are considering corresponds in the DL tradition to the 1-satisfiability of the assertion $C(a)$.

[3] For the 1-satisfiability case the problem is reduced to 1-satisfiability of certain propositional formula; and for the validity problem it is reduced to a semantic consequence problem, with a finite number of hypothesis, in the propositional fuzzy logic.

Using these two clauses it is obvious that concepts r-satisfiable in a witnessed interpretation are also r-satisfiable in a finite model; and in the case of Łukasiewicz t-norm it is well known (Hájek 2007) that first order formulas (in particular this applies to DL concepts) r-satisfiable are r-satisfiable in a witnessed interpretation.

In this paper we will restrict to the case of product t-norm. First of all, we want to provide an example which motivates why is it worth considering product t-norm in order to built fuzzy DL languages.

Example 1 (Restaurant Finder). *Consider the following $ABox$ about a restaurant finder:*

- *individuals in our KB are the restaurants:* `rest1`, `rest2`, `rest3`, `rest4`,
- *concepts in our KB are:* `Near` *and* `Good`, *with the usual intuitive interpretation,*
- *moreover, we have the following set of assertions:* $\mathcal{A} = \{\langle \texttt{rest1} : \texttt{Near} = 0.8\rangle, \langle \texttt{rest2} : \texttt{Near} = 0.6\rangle, \langle \texttt{rest3} : \texttt{Near} = 0.4\rangle, \langle \texttt{rest4} : \texttt{Near} = 0.2\rangle, \langle \texttt{rest1} : \texttt{Good} = 0.6\rangle, \langle \texttt{rest2} : \texttt{Good} = 0.7\rangle, \langle \texttt{rest3} : \texttt{Good} = 0.2\rangle, \langle \texttt{rest4} : \texttt{Good} = 0.8\rangle\}$.

In the following table we show what happens when we process the above $ABox$ by means of each one of the three basic t-norms in order to search for a restaurant that is both near and good, i.e., in which degree the four restaurants of the example are instances of the concept `Near` $*$ `Good`, *where $*$ stands for one of the basic t-norms.*

	Ł	*G*	Π
`rest1`	*0.4*	*0.6*	*0.48*
`rest2`	*0.3*	*0.6*	*0.42*
`rest3`	*0*	*0.2*	*0.08*
`rest4`	*0*	*0.2*	*0.16*

As we can see, in the case of Łukasiewicz t-norm, beyond certain distance there is no difference between restaurants, because the distance makes every value collapse to 0. *In the case of Gödel t-norm, the resulting value is not a combination of the two values at the beginning, but one of them, regardless it refers to quality or distance. Nevertheless, we consider important to have minimum t-norm expressible in the language, since it is strictly related with the hierarchy of concepts. In the product case, each restaurant that falls within the distance range we have chosen to check, has an assigned value that depends on both distance and quality. In this example product t-norm seems to be more adequate for modeling purposes than the other two basic t-norms.*

The problem of decidability of a DL based on product t-norm has already been addressed by Straccia and Bobillo in (Bobillo and Straccia 2007) and in (Bobillo and Straccia 2009), but they restrict their attention to satisfiability with respect to witnessed models. Hence, they do not consider the more general case we deal in this paper.

The aim of this paper is, in fact, to prove that if $*$ is the product t-norm, then the problem Val^* is decidable. Thus, although the first order logic is non arithmetical we will see that the $*-\mathcal{ALE}$ is much more tractable. Besides the validity problem, in this paper we also consider the problems

Sat_r^* (for every $r \in [0,1]$), but for these problems we have succeeded to prove decidability only under the additional assumption of considering certain kind of interpretations (called quasi-witnessed). We point out that in our framework we are not considering reasoning tasks over TBoxes. The proof of our result follows the same pattern than Hájek's one reducing the problem to a consequence problem in the propositional fuzzy logic (of product t-norm this time), but in this occasion we cannot use witnessed interpretations. The reason is that there are concepts, like

$$\forall R.A \sqcap \neg\forall R.(A \boxdot A) \quad \text{and} \quad \neg\forall R.A \sqcap \neg\exists R.\neg A$$

which are 1-satisfiable, but never in a witnessed interpretation (cf. (Hájek and Cintula 2006)). Thus, in order to deal with the product case, we will have to explain how to codify infinite interpretations using a finite number of propositional formulas. We will show how to do this codification for what we will call quasi-witnessed[4] interpretations, and we will see that these quasi-witnessed interpretations are indeed enough in order to deal with the product t-norm case.

2. Preliminaries

In this preliminary section we firstly introduce the syntax and semantics of $*-\mathcal{ALE}$. The syntax is the same one for every t-norm $*$, that is, it is independent of $*$. We fix a finite set of concept names and a finite set of role names, and using these fixed sets we inductively define *concepts* as follows:

$$C, D \rightsquigarrow A \mid \bot \mid \top \mid C \boxdot D \mid C \to D \mid \forall R.C \mid \exists R.C$$

where A is a concept name, C and D are concepts and R is a role name.

Although we are not considering as primitive common connectives like $\sqcap$, $\sqcup$ and $\neg$, we remind the reader that, as has been pointed out above, they are definable from the primitive ones. The following definition introduces the intended semantics behind concepts in $*-\mathcal{ALE}$.

Definition 2. Let $*$ be a t-norm and let $\Rightarrow$ be its residuum. Then, an *$*$-interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a (crisp) set $\Delta^{\mathcal{I}}$ (called the *domain* of $\mathcal{I}$) and an *interpretation function* $\cdot^{\mathcal{I}}$, which maps every concept C to a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$, every role name R to a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$ and such that, for every concepts C, D, every role name R and every element $a \in \Delta^{\mathcal{I}}$, it holds that:

$$\begin{array}{rcl}
\bot^{\mathcal{I}}(a) & = & 0 \\
\top^{\mathcal{I}}(a) & = & 1 \\
(C \boxdot D)^{\mathcal{I}}(a) & = & C^{\mathcal{I}}(a) * D^{\mathcal{I}}(a) \\
(C \to D)^{\mathcal{I}}(a) & = & C^{\mathcal{I}}(a) \Rightarrow D^{\mathcal{I}}(a) \\
(\forall R.C)^{\mathcal{I}}(a) & = & \inf\{R^{\mathcal{I}}(a,b) \Rightarrow C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}}(a) & = & \sup\{R^{\mathcal{I}}(a,b) * C^{\mathcal{I}}(b) : b \in \Delta^{\mathcal{I}}\}
\end{array}$$

Definition 3. An $*-$interpretation $\mathcal{I}$ is *quasi-witnessed* when it satisfies condition (wit$\exists$) and

(qwit$\forall$) for every concept C, every role name R and every $a \in \Delta^{\mathcal{I}}$ either $(\forall R.C)^{\mathcal{I}}(a) = 0$ or there is some $b \in \Delta^{\mathcal{I}}$ such that

$$(\forall R.C)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a,b) \Rightarrow C^{\mathcal{I}}(b).$$

In the rest of the paper we will focus on the product t-norm, which will be denoted as Π. Thus, from now on the reader can always assume that $*$ and $\Rightarrow$ are the functions from $[0,1]^2$ into $[0,1]$ defined by

$$x * y := x \cdot y \quad \text{and} \quad x \Rightarrow y := \begin{cases} 1 & \text{if } x \le y \\ y/x & \text{otherwise} \end{cases}$$

We will use the notations Sat_r and Val to denote the sets Sat_r^{Π} and Val^{Π} introduced above; and we will write QSat_r and QVal to denote the same problems restricted to quasi-witnessed interpretations. We stress that the aim of this paper is to prove the following theorem.

Theorem 4 (Product Case). *For every $r \in [0,1]$, the set QSat_r is decidable; and the set QVal is also decidable.*

First of all we notice that for every $r, s \in (0,1)$, $\mathrm{QSat}_r = \mathrm{QSat}_s$. This is an immediate consequence of the fact that for every $l \in \mathbb{R}^+$, the function $x \longmapsto x^l$ is an order isomorphism and a homomorphism of the operations $*$ and $\Rightarrow$. We will use the notation QSat_i to indicate the set of concepts that are intermediately satisfiable (i.e., 0.5 satisfiable). Therefore, in this paper we only need to deal with the sets QSat_0, QSat_1, QSat_i and QVal. Moreover, using that

- $C \in \mathrm{QSat}_0$ iff $\neg C \in \mathrm{QSat}_1$, and
- $C \in \mathrm{QSat}_i$ iff $C \sqcup \neg C \notin \mathrm{QVal}$,

it follows that it would be enough to prove that QSat_1 and QVal are decidable. It is obvious that all statements in this paragraph also hold if we replace QSat_1 with Sat_1, QSat_i with Sat_i, QSat_0 with Sat_0, and QVal with Val.

The future sections of this paper will be devoted to prove Theorem 4. The proof consists on a reduction of the problems QSat_1 and QVal to a consequence problem of the infinite-valued propositional Product Logic.

In the rest of this section we notice that Theorem 4 can be used to prove that the problem Val (without restricting to quasi-witnessed interpretations) is decidable. This is a consequence of Proposition 5. The proof of this proposition is given in an appendix because it is a technical proof based on first order fuzzy logics, with no particular relationship to fuzzy DLs.

Proposition 5. *Let C be a concept. Then,*

$$C \in \mathrm{Val} \quad \textit{iff} \quad C \in \mathrm{QVal}.$$

Theorem 6 (Product Case). *The problems Sat_i and Val are decidable.*

Proof. It follows from Theorem 4 and Proposition 5. ☐

3. The Reduction to the Propositional Case

In order to prove that r-satisfiability in a quasi-witnessed Π-interpretation is decidable we are going to codify quasi-witnessed Π-interpretations by some finite number of formulas in the propositional product logic; using this finite

[4] These interpretations, in the first order setting, were firstly considered in (Laskowski and Malekpour 2007) under the name of *closed* models. Since closed has many different meanings in logic and mathematics, we prefer to call them *quasi-witnessed* that is more informative.

codification we will not know how to recover the same initial Π-interpretation, but we will be able to build a Π-interpretation with the same associated truth value. Here it will be crucial the fact that the mappings $x \longmapsto x^i$ are isomorphisms of $[0,1]_\Pi$ (for every $i \in \omega$).

First of all, let us a fix an infinite set $\mathrm{Ind} = \{a_i : i \in \omega\}$, whose elements will be called *individuals* or *constants*. An *assertion* is any propositional combination of expressions of the forms $C(a)$ and $R(a,b)$ where C is a concept, R is a role name and a, b are individuals.

Before introducing the algorithm, we need some previous definitions.

Definition 7. 1. *Nesting degree of quantifiers* in C (or $C(a)$) is defined inductively: $nest(A) = 0$, if A is an atomic concept name; if C and D are concepts, then $nest(C \boxdot D) = nest(C \to D) = max(nest(C), nest(D))$; finally, if C is a concept and R a role name, then $nest(\forall R.C) = nest(\exists R.C) = nest(C) + 1$.

2. *Generalized atoms* are quantified concepts, i.e. concepts of the form $\forall R.C$ or $\exists R.C$, where C is a propositional combination of concepts and generalized atoms; the latter will be called *generalized atoms* of C. We will also use the term generalized atom for instances of quantified concepts, the context will clarify the precise meaning.

Definition 8 (Labelling). Let C be a concept. The *labelling function* is the function which associates to every occurrence D of a subconcept in C an element of $\mathbb{N}^{\leq k}$ (where $k = nest(C)$) defined by the conditions:

1. $l(C)$ is the empty sequence $\emptyset$,
2. if D is a propositional combination of concepts $D_1, \ldots, D_n$, then $l(D_i) := l(D)$ for every $i \leq n$.
3. if D is $\forall R.D'$ or $\exists R.D'$, then $l(D')$ is the concatenated sequence $l(D), n$, where n is the minimum number m such that the sequence $l(D), m$ has not been used to label any occurrence in C.

In order to illustrate the notions defined in the last definitions, as well as further definitions, we propose an example that will be used throughout the paper.

Example 9. *Consider the concept*

$$C := \forall R.\exists R.A \sqcap \neg\forall R.(\exists R.A \boxdot \exists R.A)$$

where A is an atomic concept. Then,

1. *concept C has nesting degree 2.*
2. *the generalized atoms in C are: $\forall R.\exists R.A$, $\forall R.(\exists R.A \boxdot \exists R.A)$ and $\exists R.A$.*
3. *the labelling function associated with occurrences in C is given by the genealogical tree*

$$\frac{\dfrac{\dfrac{A:1,1}{\exists R.A:1}}{\forall R.\exists R.A:\emptyset} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A:2,1}{\exists R.A:2} \quad \dfrac{A:2,2}{\exists R.A:2}}{\exists R.A \boxdot \exists R.A:2}}{\forall R.(\exists R.A \boxdot \exists R.A):\emptyset}}{\neg\forall R.(\exists R.A \boxdot \exists R.A):\emptyset}}{C:\emptyset}$$

Here we have used the notation $D : \sigma$ to indicate that the labelling of occurrence D is the sequence σ.

Next, for every concept C we are going to recursively associate two finite sets T_C and Y_C of assertions.

Definition 10 (Algorithm). Given a concept C_0, we construct finite sets T_{C_0} and Y_{C_0} of assertions. The construction takes steps $0, \ldots, n$ where n is the nesting degree of the concept C_0. At each step some generalized atoms are processed; and at each step we add some new constants from Ind and some new formulas to T_{C_0} and Y_{C_0} and we transfer some assertions of concepts for processing in the next step. The assertions produced in step i will have nesting degree $n-i$; after step n is completed the algorithm stops.

At step 0, we simply transfer the assertion $C_0(d)$ to be further processed in step 1; and we say that constant d has level 0. For $i > 0$, step i selects the generalized atoms in formulas transferred from step $i-1$ and processes them. We know that the generalized atoms just selected have the form $QR.C(d_\sigma)$, where $Q \in \{\forall, \exists\}$, R is a role, C a concept with nesting degree $\leq n-i$, d_σ is a constant produced in the previous step and σ is the label of the generalized atom we are considering. For each generalized atom α, at step i we firstly do the following:

- If α is $\forall R.C(d_\sigma)$, then produce a new constant $d_{\sigma,n}$ and add to T_{C_0} the assertion
$$(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n}))) \sqcup \neg\forall R.C(d_\sigma)$$
- If α is $\exists R.C(d_\sigma)$, then produce a new constant $d_{\sigma,n}$ and add to T_{C_0} the assertion:
$$(R(d_\sigma, d_{\sigma,n}) \boxdot C(d_{\sigma,n})) \equiv \exists R.C(d_\sigma)$$

We will say that $d_{\sigma,n}$ is a *constant associated to* R, d_σ. Now, we consider each α of the present step and do the following:

- If α is $(\forall R.C)(d_\sigma)$ and $d_{\sigma,m}$ is any constant associated to R, d_σ, then add to T_{C_0} the assertion
$$\forall R.C(d_\sigma) \to (R(d_\sigma, d_{\sigma,m}) \to C(d_{\sigma,m}))$$
- If α is $(\exists R.C)(d_\sigma)$ and $d_{\sigma,m}$ is any constant associated to R, d_σ, then add to T_{C_0} the assertion
$$(R(d_\sigma, d_{\sigma,m}) \boxdot C(d_{\sigma,m})) \to \exists R.C(d_\sigma)$$
- If α is $(\forall R.C)(d_\sigma)$, then add to Y_{C_0} the assertion
$$\neg\forall R.C(d_\sigma) \boxdot (R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n}))$$

Example 11. *Following Definition 10, the assertions belonging to T_C are:*

- $(\forall R.\exists R.A(d) \equiv (R(d,d_1) \to \exists R.A(d_1))) \sqcup \neg\forall R.\exists R.A(d)$,
- $(\forall R.(\exists R.A \boxdot \exists R.A)(d) \equiv (R(d,d_2) \to (\exists R.A \boxdot \exists R.A)(d_2))) \sqcup \neg\forall R.(\exists R.A \boxdot \exists R.A)(d)$,
- $\forall R.\exists R.A(d) \to (R(d,d_2) \to A(d_2))$,
- $\forall R.(\exists R.A \boxdot \exists R.A)(d) \to (R(d,d_1) \to (\exists R.A \boxdot \exists R.A)(d_1))$,
- $\exists R.A(d_1) \equiv (R(d_1, d_{1,1}) \boxdot A(d_{1,1}))$,
- $\exists R.A(d_2) \equiv (R(d_2, d_{2,1}) \boxdot A(d_{2,1}))$,

- $\exists R.A(d_2) \equiv (R(d_2, d_{2,2}) \boxdot A(d_{2,2}))$,
- $(R(d_2, d_{2,2}) \boxdot A(d_{2,2})) \to \exists R.A(d_2)$,
- $(R(d_2, d_{2,1}) \boxdot A(d_{2,1})) \to \exists R.A(d_2)$.

While assertions belonging to Y_C are:

- $\neg\forall R.\exists R.A(d) \boxdot (R(d, d_1) \to \exists R.A(d_1))$,
- $\neg\forall R.(\exists R.A \boxdot \exists R.A)(d) \boxdot (R(d, d_2) \to (\exists R.A \boxdot \exists R.A)(d_2))$.

As it is said above our aim is to reduce our problem to one in the propositional Product Logic. Here we will consider this propositional logic using as variables the set

$$Prop := \{p_{R(a,b)} : R \text{ is a role name and } a, b \in \text{Ind}\} \cup$$
$$\{p_{C(a)} : C \text{ atomic or quantified concept and } a \in \text{Ind}\}.$$

We stress that we are taking a concrete fix set as variables. Nevertheless, for a particular concept C it is clear that a finite subset $Prop_C$ of $Prop$ would be enough. Using that all concepts are indeed propositional combinations of expressions of the form $C(a)$ and $R(a, b)$, the following definition is meaningful. This definition tells us that we can look at assertions as propositional formulas with variables in $Prop$.

Definition 12. The map pr associates to every assertion a formula in the propositional logic (with the variables given above) according to the following clauses:

1. $pr(C(a)) = p_{A(a)}$ if C is an atomic or a quantified concept,
2. $pr(R(a, b)) = p_{R(a,b)}$ if R is a role name and $a, b \in \text{Ind}$,
3. $pr(\bot(a)) = \bot$,
4. $pr(\top(a)) = \top$
5. $pr((C \boxdot D)(a)) = pr(C(a)) \odot pr(D(a))$,
6. $pr((C \to D)(a)) = pr(C(a)) \to pr(D(a))$.

If T is a set of assertions, then $pr(T)$ is $\{pr(\alpha) \mid \alpha \in T\}$.

Example 13. *If $T_C$is the set defined in the previous example, then, following Definition 12, propositional formulas belonging to $pr(T)$ are:*

- $(p_{\forall R.\exists R.A(d)} \equiv (p_{R(d,d_1)} \to p_{\exists R.A(d_1)})) \vee \neg p_{\forall R.\exists R.A(d)}$,
- $(p_{\forall R.(\exists R.A \odot \exists R.A)(d)} \equiv (p_{R(d,d_2)} \to (p_{\exists R.A \odot \exists R.A)(d_2)})) \vee \neg p_{\forall R.(\exists R.A \odot \exists R.A)(d)}$,
- $p_{\forall R.\exists R.A(d)} \to (p_{R(d,d_2)} \to p_{A(d_2)})$,
- $p_{\forall R.(\exists R.A \odot \exists R.A)(d)} \to (p_{R(d,d_1)} \to p_{(\exists R.A \odot \exists R.A)(d_1)})$,
- $p_{\exists R.A(d_1)} \equiv (p_{R(d_1,d_{1,1})} \odot p_{A(d_{1,1})})$,
- $p_{\exists R.A(d_2)} \equiv (p_{R(d_2,d_{2,1})} \odot p_{A(d_{2,1})})$,
- $p_{\exists R.A(d_2)} \equiv (p_{R(d_2,d_{2,2})} \odot p_{A(d_{2,2})})$,
- $(p_{R(d_2,d_{2,2})} \odot p_{A(d_{2,2})}) \to p_{\exists R.A(d_2)}$,
- $(p_{R(d_2,d_{2,1})} \odot p_{A(d_{2,1})}) \to p_{\exists R.A(d_2)}$.

On the other hand, propositional formulas belonging to $pr(Y_C)$ are:

- $\neg p_{\forall R.\exists R.A(d)} \odot (p_{R(d,d_1)} \to p_{\exists R.A(d_1)})$,
- $\neg p_{\forall R.(\exists R.A \odot \exists R.A)(d)} \odot (p_{R(d,d_2)} \to p_{(\exists R.A \odot \exists R.A)(d_2)})$.

The next and crucial step in the proof is the following result. We leave the proofs of each one of the directions for the future two sections.

Proposition 14. *Let C_0 be a concept, and let T_{C_0} and Y_{C_0} be the two finite sets associated by Definition 10. For every $r \in [0, 1]$, the following statements are equivalent:*

1. *C_0 is satisfiable with truth value r in a quasi-witnessed Π-interpretation,*
2. *there is some propositional evaluation e over the set $Prop$ such that $e(pr(C(d_0))) = r$, $e[pr(T_{C_0})] = 1$, and $e[\psi] \neq 1$ for every $\psi \in pr(Y_{C_0})$.*

From now on we will say that a propositional evaluation e is *quasi-witnessing relatively to C_0* (*quasi-witnessing*, for short) when it satisfies that $e[pr(T_{C_0})] = 1$, and $e[\psi] \neq 1$ for every $\psi \in pr(Y_{C_0})$.

As a consequence of this last proposition we are now able to prove Theorem 4. This is so because by Proposition 14 we know that

- $C \in \text{QSat}_1$ iff $\bigvee pr(Y_{C_0})$ is not derivable, in the propositional product logic, from the set $\{pr(C(d_0))\} \cup pr(T_{C_0})$,
- $C \in \text{QVal}$ iff $pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$ is derivable, in the propositional product logic, from the set $pr(T_{C_0})$.

Hence, we have a reduction of these problems to the semantic consequence problem, with a finite number of hypothesis, in the propositional product logic. This problem can be formalized as the problem of deciding, given two propositional formulas φ and ψ, whether ψ is a semantic consequence of φ, i.e., whether, each propositional evaluation which gives value 1 to φ, also gives value 1 to ψ. In (Hájek 2006, Theorem 3) it is proved that such problem is in $PSPACE$ for the expansion of product logic with truth constants, but, since a formula without truth constants can be considered as a formula of the expanded language in which do not appear truth constants, this result also holds for the product logic without truth constants. Thus, the proof of Proposition 14 is the only missing step in order to prove Theorem 4.

4. From DL interpretations to propositional evaluations

The purpose of this section is to show the downwards implication of Proposition 14. Let us assume that for a given concept C_0, there is a quasi-witnessed Π-interpretation $\mathcal{I}$ and an object a such that $C_0^{\mathcal{I}}(a) = r$ for some $r \in [0, 1]$. The following definition tells us a way to obtain a propositional evaluation satisfying the requirements in Proposition 14.

Definition 15. Let $\mathcal{I}$ be a quasi-witnessed interpretation, a an object of the domain and C_0 a concept. Let us consider T_{C_0}, Y_{C_0} as the sets of assertions obtained from the concept C_0 by applying Definition 10. We assume that the individual a_0 has been interpreted in $\mathcal{I}$ as the object a; for each step, assume that constants in previous steps have been interpreted in $\mathcal{I}$. For each generalized atom α processed in that step, do the following:

($\forall$1) If $\alpha = \forall R.C(d_\sigma)$ and there exists $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then interpret the constant $d_{\sigma,n}$ as u (calling the expansion of $\Delta^{\mathcal{I}}$ by these constants again $\Delta^{\mathcal{I}}$).

($\forall 2$) If $\alpha = \forall R.C(d_\sigma)$ and there is no $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then choose an element $u \in \Delta^{\mathcal{I}}$ such that $0 < R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, u) \Rightarrow C^{\mathcal{I}}(u) < 1$. Such an element exists since, being $\mathcal{I}$ a quasi-witnessed interpretation, we have, on the one hand, that, for each $u \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, u) \Rightarrow C^{\mathcal{I}}(u) > 0$ and, on the other hand, if there was no element $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, u) \Rightarrow C^{\mathcal{I}}(u) < 1$, then $\inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d) \Rightarrow C^{\mathcal{I}}(d)\} = 1 = R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, u) \Rightarrow C^{\mathcal{I}}(u)$ against the supposition. Once chosen the element u, interpret the constant $d_{\sigma,n}$ as u (calling the expansion of $\Delta^{\mathcal{I}}$ by these constants again $\Delta^{\mathcal{I}}$).

($\exists$) If $\alpha = \exists R.C(d_\sigma)$, then choose an element $u \in \Delta^{\mathcal{I}}$ witnessing α and interpret the constant $d_{\sigma,n}$ as u (calling the expansion of $\Delta^{\mathcal{I}}$ by these constants again $\Delta^{\mathcal{I}}$).

Finally, for every generalized atom and every atomic formula α, occurring in T, define $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$.

Using and modifying an example reported in (Bobillo and Straccia 2009), we provide the following instance of the above definition.

Example 16. *Consider the interpretation $\mathcal{I}$ such that:*

1. $\Delta^{\mathcal{I}} = \{a, b, c, e, f\} \cup \{e_i \mid i \in \omega \backslash \{0\}\}$,
2. *there is a binary relation r such that $r(b, c) = r(e, f) = 1$, $r(a, b) = r(a, e) = 0.5$, $r(a, e_i) = 0.5^i$, and $R(x, y) = 0$, when x, y is any other pair of elements of the domain.*
3. *there is a unary predicate s such that $s(c) = s(f) = 0.5$ and $s(x) = 0$ for any other element x of the domain.*

So, if we take $R^{\mathcal{I}} = r$, $A^{\mathcal{I}} = s$, $d^{\mathcal{I}} = a$, $d^{\mathcal{I}}_1 = b$, $d^{\mathcal{I}}_{1,1} = c$, $d^{\mathcal{I}}_2 = e$ and $d^{\mathcal{I}}_{2,1} = d^{\mathcal{I}}_{2,2} = f$, then it is easy to check that:

1. *$\mathcal{I}$ is a quasi-witnessed model of concept C,*
2. *$e_{\mathcal{I}}(pr(\varphi)) = 1$, for each $\varphi \in T_C$,*
3. *$e_{\mathcal{I}}(pr(\psi)) < 1$, for each $\psi \in Y_C$.*

With the following Lemma and Proposition, we are going to prove that all propositional evaluations obtained in this way are quasi-witnessing.

Lemma 17. *Let $\mathcal{I}$ be a quasi-witnessed interpretation, C_0 a concept, and let us consider T_{C_0}, Y_{C_0} as the sets of assertions obtained from the concept C_0 by applying Definition 10. Then, the propositional evaluation $e_{\mathcal{I}}$ is quasi-witnessing relatively to C_0.*

Proof. We will show the result considering, case by case, the five kinds of proposition we can find in $pr(T_{C_0})$ and $pr(Y_{C_0})$.

1. Consider the assertion $(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n}))) \sqcup \neg\forall R.C(d_\sigma)$, then:

   ($\forall 1$) if, following Definition 15, we have interpreted the constant $d_{\sigma,n}$ as an element $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then we have that $e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n}))) \sqcup \neg\forall R.C(d_\sigma))) = (e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) \equiv (e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,n}))) \to e_{\mathcal{I}}(pr(C(d_{\sigma,n}))))) \vee \neg e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = ((\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) \equiv (R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d^{\mathcal{I}}_{\sigma,n}) \Rightarrow C^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma,n}))) \vee \neg(\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) = (\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) \equiv (R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d^{\mathcal{I}}_{\sigma,n}) \Rightarrow C^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma,n})) = 1$.

   ($\forall 2$) if there is no $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then, since $\mathcal{I}$ is a quasi-witnessed interpretation, $e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n}))) \sqcup \neg\forall R.C(d_\sigma))) = (e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) \equiv e_{\mathcal{I}}((pr(R(d_\sigma, d_{\sigma,n}))) \to e_{\mathcal{I}}(pr(C(d_{\sigma,n}))))) \vee \neg e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = ((\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) \equiv (R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d^{\mathcal{I}}_{\sigma,n}) \Rightarrow C^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma,n}))) \vee \neg(\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) = \neg(\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) = 1$. In every case we have that $e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n}))) \vee \neg\forall R.C(d_\sigma))) = 1$.

2. Consider the assertion $\exists R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \boxdot C(d_{\sigma,n}))$. Then, by Definition 15, we have that $e_{\mathcal{I}}(pr(\exists R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,n}) \boxdot C(d_{\sigma,n})) = e_{\mathcal{I}}(pr(\exists R.C(d_\sigma))) \equiv e_{\mathcal{I}}((pr(R(d_\sigma, d_{\sigma,n}))) \odot e_{\mathcal{I}}(pr(C(d_{\sigma,n})))) = \exists R.C^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) \equiv (R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d^{\mathcal{I}}_{\sigma,n}) \cdot C^{\mathcal{I}}(d^{\mathcal{I}}_{\sigma,n})) = 1$.

3. Consider the assertion $\forall R.C(d_\sigma) \to (R(d_\sigma, d_{\sigma,m}) \to C(d_{\sigma,m}))$. Since $(\forall R.C(d_\sigma))^{\mathcal{I}} = \inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then, by Definition 15 we have that $e_{\mathcal{I}}(pr(\forall R.C(d_\sigma) \to (R(d_\sigma, d_{\sigma,m}) \to C(d_{\sigma,m})))) = e_{\mathcal{I}}(pr(\forall R.C(d_\sigma)) \to (e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m})) \to e_{\mathcal{I}}(pr(C(d_{\sigma,m}))) = (\forall R.C)^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) \Rightarrow (R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d^{\mathcal{I}}_{\sigma,m}) \Rightarrow C(d^{\mathcal{I}}_{\sigma,m})) = 1$.

4. Consider the assertion $(R(d_\sigma, d_{\sigma,m}) \boxdot C(d_{\sigma,m})) \to \exists R.C(d_\sigma)$. Since $(\exists R.C(d_\sigma))^{\mathcal{I}} = \sup_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d) \cdot C^{\mathcal{I}}(d)\}$, then, by Definition 15 we have that $e_{\mathcal{I}}(pr((R(d_\sigma, d_{\sigma,m}) \to C(d_{\sigma,m}))) \to \exists R.C(d_\sigma)) = (e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,m})) \cdot e_{\mathcal{I}}(pr(C(d_{\sigma,m}))) \Rightarrow e_{\mathcal{I}}(pr(\exists R.C(d_\sigma)) = (R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d^{\mathcal{I}}_{\sigma,m}) \cdot C(d^{\mathcal{I}}_{\sigma,m})) \Rightarrow \exists R.C^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) = 1$.

5. Consider the assertion $\neg\forall R.C(d_\sigma) \boxdot (R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n}))$, then:

   ($\forall 1$) if, following Definition 15, we have interpreted the constant $d_{\sigma,n}$ as an element $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then we have that if, on the one hand, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma))) = 1$, then $e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = 0$ and, therefore, by Definition 10, $e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n}))) = 0$. Hence $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma) \boxdot (R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n})))) = 0 < 1$. If, on the other hand, $e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n}))) = 1$, then, by assumption, $e_{\mathcal{I}}(pr(\forall R.C(d_\sigma))) = 1$ and, therefore, again, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma) \boxdot (R(d_\sigma, d_{\sigma,n}) \to C(d_{\sigma,n})))) = 0 < 1$.

   ($\forall 2$) if there is no $u \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, u) \Rightarrow C^{\mathcal{I}}(u) = \inf_{d \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d^{\mathcal{I}}_\sigma, d) \Rightarrow C^{\mathcal{I}}(d)\}$, then, since $\mathcal{I}$ is a quasi-witnessed interpretation, we have that, necessarily, $\forall R^{\mathcal{I}}.C^{\mathcal{I}}(d^{\mathcal{I}}_\sigma) = 0$ and, hence, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma))) = 1$. However, since, by Definition 15, , we have interpreted the constant

$d_{\sigma,n}$ as an element $u \in \Delta^{\mathcal{I}}$ such that $0 < R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) < 1$ and, therefore, we have that $e_{\mathcal{I}}(pr(R(d_\sigma, d_{\sigma,n}) \rightarrow C(d_{\sigma,n}))) < 1$. So, $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma) \boxdot (R(d_\sigma, d_{\sigma,n}) \rightarrow C(d_{\sigma,n})))) < 1$.

Hence, for every proposition $pr(\varphi) \in pr(T)$, it holds that $e_{\mathcal{I}}(pr(\varphi)) = 1$ and for every proposition $pr(\psi) \in pr(Y)$, it holds that $e_{\mathcal{I}}(pr(\psi)) < 1$ and, therefore, $e_{\mathcal{I}}$ is a quasi-witnessing propositional evaluation. □

Proposition 18. *Let $\mathcal{I}$ be a quasi-witnessed interpretation, $C_0(a_0)$ a Π-$\mathcal{ALE}$-assertion and T, Y the sets of assertions produced from $C_0(a_0)$ applying Definition 10, then, for every $\alpha \in T \cup Y$, it holds that $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$.*

Proof. We will prove the Lemma by induction on the costruction of α.

1. If α is an atomic formula, it is straightforward from Definition 15.
2. If α is a generalized atom, it is straightforward from Lemma 17.
3. If α is of the form $\delta \star \gamma$ where δ, γ are either atomic formulas or generalized atoms and $\star$ is a propositional operator, suppose, by inductive hypothesis, that $e_{\mathcal{I}}(pr(\delta)) = \delta^{\mathcal{I}}$ and $e_{\mathcal{I}}(pr(\gamma)) = \gamma^{\mathcal{I}}$. Hence, $e_{\mathcal{I}}(pr(\alpha)) = e_{\mathcal{I}}(pr(\delta \star \gamma)) = e_{\mathcal{I}}(pr(\delta)) \star e_{\mathcal{I}}(pr(\gamma)) = \delta^{\mathcal{I}} \star \gamma^{\mathcal{I}} = (\delta \star \gamma)^{\mathcal{I}} = \alpha^{\mathcal{I}}$.

Hence, for every proposition $pr(\alpha)$ in $pr(T \cup Y)$, it holds that $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$. In particular, $e_{\mathcal{I}}(pr(C_0(a_0))) = C_0^{\mathcal{I}}(a_0^{\mathcal{I}})$. □

This finishes the proof of the downwards implication of Proposition 14.

5. From propositional evaluations to DL interpretations

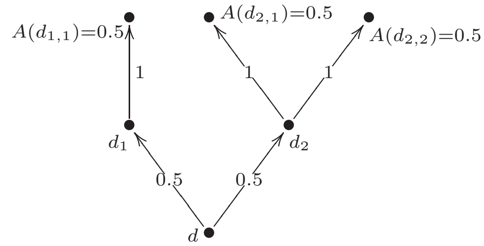
The aim of this section is to prove the upwards implication of Proposition 14. Let us assume that there is a propositional evaluation quasi-witnessing relatively to C_0 such that $e(pr(C_0(d))) = r$ for some $r \in [0, 1]$. First of all, we provide a way to obtain a quasi-witnessed Π-interpretation from a quasi-witnessing propositional evaluation with the above features.

Definition 19. Let α be an assertion, T and Y the sets of concepts and axioms produced from α applying Definition 10, let $pr(T)$, $pr(Y)$ be the sets of propositions obtained by applying Definition 12 and let e be a quasi-witnessing propositional evaluation. Then we define the witnessed part $\mathcal{I}_e^w$ of our first order interpretation $\mathcal{I}_e$ as follows:

1. $\Delta^{\mathcal{I}_e^w}$ is the set of all constants d_σ occurring in formulas of T.
2. For each atomic concept C, let:
   (a) $C^{\mathcal{I}_e^w}(d_\sigma) = e(pr(C(d_\sigma)))$, where $\sigma = l(C)$, if $pr(C(d_\sigma))$ occurs in $pr(T)$,
   (b) $C^{\mathcal{I}_e^w}(d_\sigma) = 0$, otherwise.
3. For each role R let:
   (a) $R^{\mathcal{I}_e^w}(d_\sigma, d_{\sigma,n}) = e(pr(R(d_\sigma, d_{\sigma,n})))$, if $pr(R(d_\sigma, d_{\sigma,n}))$ occurs in $pr(T)$,
   (b) $R^{\mathcal{I}_e^w}(d_\sigma, d_{\sigma,n}) = 0$, otherwise.

In order to illustrate Definition 19, we provide an example of the witnessed interpretation arising from $pr(T_C)$ and $pr(Y_C)$.

Example 20. *Let e be a propositional evaluation such that $p_{R(d,d_1)} = p_{R(d,d_2)} = 0.5$, $p_{R(d_1,d_{1,1})} = p_{R(d_2,d_{2,1})} = p_{R(d_2,d_{2,2})} = 1$, $p_{A(d_{1,1})} = p_{A(d_{2,1})} = p_{A(d_{2,2})} = 0.5$. As we have seen in the previous section, this is indeed a quasi-witnessing propositional evaluation. Moreover, following Definition 19, we obtain the following interpretation:*



We point out that this interpretation, however, is not a model of the concept C.

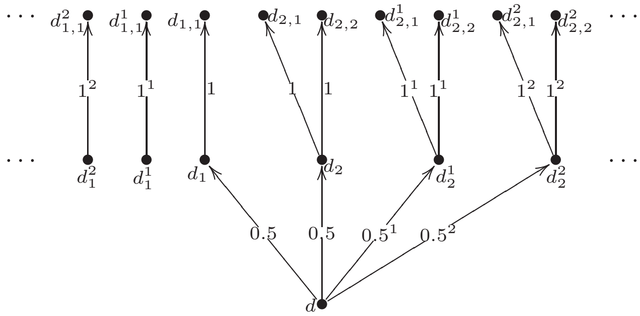
The structure defined in Definition 19 is a witnessed interpretation which would be enough in case we were only interested on witnessed interpretations. But in order to encompass all quasi-witnessed Π-interpretations we need the following extension of the above interpretation.

Definition 21. Let α be an assertion, T and Y the sets of first order formulas produced from α applying Definition 10, let $pr(T)$, $pr(Y)$ be the sets of propositions obtained by applying Definition 12 and let e be a quasi-witnessing propositional evaluation; finally let $\mathcal{I}_e^w$ be the interpretation defined in Definition 19. Then we define the first order interpretation $\mathcal{I}_e$ as the following expansion of $\mathcal{I}_e^w$:

1. The domain $\Delta^{\mathcal{I}_e}$ is obtained by adding to $\Delta^{\mathcal{I}_e^w}$ an infinite set of new individuals $\{d_\sigma^i | i \in \omega\backslash\{0\}\}$, for each $d_\sigma \in \Delta^{\mathcal{I}_e^w}$, but not for d.
2. if C is an atomic concept, and $pr(C(d_\sigma^i))$ occurs in $pr(T)$, then $C^{\mathcal{I}_e}(d_\sigma^i) = (C^{\mathcal{I}_e}(d_\sigma))^i$,
3. For each role R:
   (a) if R appears in an universally quantified formula, then:
      i. if $e(pr(\forall R.C(d_\sigma))) \neq e(pr(R(d_\sigma, d_{\sigma,n}) \rightarrow C(d_\alpha)))$, then:
         A. $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}^i) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^i$, for every $i \in \omega\backslash\{0\}$,
         B. $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^j$, for every $i, j \in \omega\backslash\{0\}$,
      ii. if $e(pr(\forall R.C(d_\sigma))) = e(pr(R(d_\sigma, d_{\sigma,n}) \rightarrow D(d_{\sigma,n})))$, then $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^j$, for every $i, j \in \omega\backslash\{0\}$, if $i = j$ and $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = 0$, otherwise,
   (b) if R appears in an existentially quantified formula, then $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^j$, for every $i, j \in \omega\backslash\{0\}$, if $i = j$ and $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,n}^j) = 0$, otherwise.

In order to illustrate Definition 21, we provide an example of the quasi-witnessed interpretation arising from $pr(T_C)$ and $pr(Y_C)$.

Example 22. *Let* e *be the same propositional evaluation as in the previous example, then, following Definition 19, we obtain the following interpretation:*



In this case it is worth pointing out that this interpretation is indeed a model of C.

Lemma 23. *Let* $D(d_\sigma) \in Sub(C)$ *and* e *a quasi-witnessing propositional evaluation, then, for each* $i \in \omega\backslash\{0\}$*, it holds that* $D^{\mathcal{I}_e}(d^i_\sigma) = (D^{\mathcal{I}_e}(d_\sigma))^i$.

Proof. The proof is by induction on the nesting degree of C.

(0) An assertion with nesting degree equal to 0 is either an atomic concept or a propositional combination of atomic concepts:

1. If C is an atomic concept, then it is straightforward from Definition 21.
2. Let $C = E \star F$, where E, F are atomic concepts and $\star \in \{\rightarrow, \boxdot\}$. Suppose, by inductive hypothesis, that the claim holds for two concepts E, F, then:
$$\begin{array}{rcl}(E^{\mathcal{I}_e} \star F^{\mathcal{I}_e})(d^i_\sigma) & = & E^{\mathcal{I}_e}(d^i_\sigma) \star F^{\mathcal{I}_e}(d^i_\sigma)\\ & = & (E^{\mathcal{I}_e}(d_\sigma))^i \star (F^{\mathcal{I}_e}(d_\sigma))^i\\ & = & (E^{\mathcal{I}_e}(d_\sigma) \star F^{\mathcal{I}_e}(d_\sigma))^i\\ & = & (E^{\mathcal{I}_e} \star F^{\mathcal{I}_e}(d_\sigma))^i\end{array}$$

(k+1) Let $D(d_\sigma)$ be a generalized atom with nesting degree equal to $k+1$ and suppose, by inductive hypothesis, that, for each generalized atom $E(d_{\sigma,n})$ with nesting degree equal to k, it holds that $E^{\mathcal{I}_e}(d^i_{\sigma,n}) = (E^{\mathcal{I}_e}(d_{\sigma,n}))^i$, then:

1. If $D(d_\sigma) = \exists R.E(d_\sigma)$, then, by Definition 21, $D^{\mathcal{I}_e}(d^i_\sigma) = \sup_{d\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d^i_\sigma, d) \cdot E^{\mathcal{I}_e}(d) = R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,n}) \cdot E^{\mathcal{I}_e}(d^i_{\sigma,n})$ and, by inductive hypothesis, Definition 10 and Definition 21, $R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,n}) \cdot E^{\mathcal{I}_e}(d^i_{\sigma,n}) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^i \cdot (E^{\mathcal{I}_e}(d_{\sigma,n}))^i = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \cdot E^{\mathcal{I}_e}(d_{\sigma,n}))^i = (D^{\mathcal{I}_e}(d_\sigma))^i$.
2. If $D(d_\sigma) = \forall R.E(d_\sigma)$, and $e(pr(\forall R.E(d_\sigma))) = (R(d_\sigma, d_{\sigma,n}) \rightarrow E(d_{\sigma,n}))$, then, by Definition 21, $E^{\mathcal{I}_e}(d^i_\sigma) = \inf_{d\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d^i_\sigma, d) \Rightarrow E^{\mathcal{I}_e}(d) = R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,n}) \Rightarrow E^{\mathcal{I}_e}(d^i_{\sigma,n})$ and, by inductive hypothesis, Definition 10 and Definition 21, $R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,n}) \Rightarrow E^{\mathcal{I}_e}(d^i_{\sigma,n}) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}))^i \Rightarrow (E^{\mathcal{I}_e}(d_{\sigma,n}))^i = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow E^{\mathcal{I}_e}(d_{\sigma,n}))^i = (D^{\mathcal{I}_e}(d_\sigma))^i$.
3. If $D(d_\sigma) = \forall R.E(d_\sigma)$, and $e(pr(\forall R.E(d_\sigma))) \neq (R(d_\sigma, d_{\sigma,n}) \rightarrow E(d_{\sigma,n}))$, then, by Definition 21, $D^{\mathcal{I}_e}(d_\sigma) = 0$ and, therefore, by Definition 21, $D^{\mathcal{I}_e}(d^i_\sigma) = \inf_{d\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d^i_\sigma, d) \Rightarrow E^{\mathcal{I}_e}(d) = \inf_{j\in\omega\backslash\{0\}}\{R^{\mathcal{I}_e}(d^i_\sigma, d^j_{\sigma,n}) \Rightarrow E^{\mathcal{I}_e}(d^j_{\sigma,n}) = 0 = (D^{\mathcal{I}_e}(d_\sigma))^i$. □

Proposition 24. *Let* e *be a quasi-witnessing propositional evaluation, then, for every assertion* α, $e(pr(\alpha)) = \alpha^{\mathcal{I}_e}$.

Proof. The proof is by induction on the nesting degree of α.

(0) An assertion with nesting degree equal to 0 is either an atomic concept or a propositional combination of atomic concepts:

1. If α is an atomic concept, then it is straightforward from Definition 19.
2. Let $\alpha = C \star D$, where C, D are concepts and $\star \in \{\rightarrow, \boxdot\}$. Suppose that the inductive hypothesis holds for two concepts C, D, then, by Definition 12 we have that, for each propositional operator $\star$:
$$\begin{array}{rcl}(C \star D)^{\mathcal{I}_e} & = & C^{\mathcal{I}_e} \star D^{\mathcal{I}_e}\\ & = & e(pr(C)) \star e(pr(D))\\ & = & e(pr(C) \star pr(D))\\ & = & e(pr(C \star D))\end{array}$$

(k+1) Let α be a generalized atom with nesting degree equal to $k+1$ and suppose, by inductive hypothesis, that, for each generalized atom β with nesting degree $\leq n$, occurring within the scope of the quantifier of α, it holds that $e(pr(\beta)) = \beta^{\mathcal{I}_e}$.

1. If $\alpha = \exists R.C(d_\sigma)$, then, by Definition 10 we have that $e(pr(\alpha)) = e(pr(R(d_\sigma, d_{\sigma,n}) \boxdot C(d_{\sigma,n})))$ and, by Definition 19 and inductive hypothesis, we have that $e(pr(R(d_\sigma, d_{\sigma,n}) \boxdot C(d_{\sigma,n}))) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \cdot C^{\mathcal{I}_e}(d_{\sigma,n})$. Let $d \in \Delta^{\mathcal{I}_e}$ be any constant different from $d_{\sigma,n}$, then either d is associated to to R, a or not. In the first case, since, by Definition 10, $e(pr(R(d_\sigma, d) \boxdot C(d))) \rightarrow e(pr(\alpha)) = 1$, then $R^{\mathcal{I}_e}(d_\sigma, d) \cdot C^{\mathcal{I}_e}(d) \leq e(pr(\alpha))$. In the second case, by Definition 19, $R^{\mathcal{I}_e}(d_\sigma, d) \cdot C^{\mathcal{I}_e}(d) = 0 \cdot C^{\mathcal{I}_e}(d) = 0 \leq e(pr(\alpha))$. So, in each case, $e(pr(\alpha)) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \cdot C^{\mathcal{I}_e}(d_{\sigma,n}) = \sup_{d\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d_\sigma, d) \cdot C^{\mathcal{I}_e}(d)\} = \alpha^{\mathcal{I}_e}$.
2. If $\alpha = \forall R.C(a)$ and $e(pr(\alpha)) = e(pr(R(d_\sigma, d_{\sigma,n}) \rightarrow C(d_{\sigma,n})))$, then, by Definition 19 and inductive hypothesis, we have that $e(pr(\alpha)) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n})$. Let $d \in \Delta^{\mathcal{I}_e}$ be any constant different from $d_{\sigma,n}$, then either d is associated to R, a or not. In the first case, since, by Definition 10, $e(pr(\alpha)) \rightarrow e(pr(R(d_\sigma, d) \rightarrow C(d))) = 1$, then $e(pr(\alpha)) \leq R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d)$. In the second case, by Definition 21, $R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d) = 0 \Rightarrow C^{\mathcal{I}_e}(d) = 1 \geq e(pr(\alpha))$. So, in each case, $e(pr(\alpha)) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n}) = \inf_{d\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d_\sigma, d) \Rightarrow C^{\mathcal{I}_e}(d)\} = \alpha^{\mathcal{I}_e}$.
3. If $\alpha = \forall R.C(a)$ and $e(pr(\alpha)) \neq e(pr(R(d_\sigma, d_{\sigma,n}) \rightarrow C(d_{\sigma,n})))$, then, by Definition 10 we have that $0 = e(pr(\alpha))$ and, by Definition 19 and inductive hypothesis, we have that $e(pr(\alpha)) \neq R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n})$. Again by Definition 10 (look at the set Y) we have that $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n}) < 1$ and, by the above assumptions, we have that $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n}) > 0$. Since, by Lemma

23, we have that, for each $i \in \omega\backslash\{0\}$, $R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d^i_{\sigma,n}) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,n}))^i$, then $e(pr(\alpha)) = 0 = \inf_{i\in\omega\backslash\{0\}}\{(R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,n}) \to C^{\mathcal{I}_e}(d_{\sigma,n}))^i\} = \inf_{i\in\omega\backslash\{0\}}\{R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,n}) \to C^{\mathcal{I}_e}(d^i_{\sigma,n})\} = \inf_{d\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d_\sigma, d) \to C^{\mathcal{I}_e}(d)\} = \alpha^{\mathcal{I}_e}$.

The result is straightforward for propositional combinations of atomic concepts and generalized atoms with nesting degree equal to $k+1$.

In particular, $e(pr(C_0(a_0))) = C_0^{\mathcal{I}_e}(a_0)$. □

This finishes the last step in the proof of Proposition 14, and so the last step in the proof of Theorem 4.

6. Further remarks and conclusions

In this paper we have proved that Val is decidable. As a consequence we get that the *positive satisfiability* problem Sat_+ is also decidable, because C is positive satisfiable iff $\neg C$ is not valid.

Among the problems considered in this paper we point out that it remains open whether Sat_1 is decidable or not. As a consequence of Theorem 4 it would be enough to prove that

$$C \in \mathrm{Sat}_1 \qquad \text{iff} \qquad C \in \mathrm{QSat}_1,$$

in order to conclude that Sat_1 is decidable.

There are also several important open questions that have not been addressed in this paper. First of all, it is still unknown a characterization of the computational complexity of the problem Val (here proved to be decidable). We notice that the reduction here considered is an exponential one, and so not helping for complexity issues. Another important open question is whether reasoning in this logic using TBoxes is decidable or not.

The crucial step in the proof we have given is Proposition 14, and the proof really depends on the fact of considering the product t-norm. On the other hand, in (Hájek 2005) it was seen that the analogous of this result based on witnessed interpretations is true for every t-norm. Thus, an interesting question here is whether this Proposition 14 holds or not for an arbitrary t-norm.

Finally, it is interesting to point out the minimum t-norm $*$ is the only basic t-norm for which it is unknown whether the problem Val^* is decidable or not. A positive answer to this problem will allow to attempts to generalize the decidability of the three basic t-norms to an arbitrary t-norm (we remind that they are obtained as ordinal sums of the three basic ones), but up to now this question remains open.

Funding

The authors thank Eurocores (LOMOREVI Eurocores Project FP006/FFI2008-03126-E/FILO), Spanish Ministry of Education and Science (project MULOG2 TIN2007-68005-C04-01, project ARINF TIN2009-14704-C03-03, project Agreement Techologies CONSOLIDER CSD 2007-0022), and Catalan Government (2009SGR-1433/34). Marco Cerami has been supported by grant JAEPredoc, n.074 of CSIC.

Acknowledgments

Special acknowledgments are due to Lluís Godo for his comments.

Appendix: Proof of Proposition 5

In this appendix we prove Proposition 5. Indeed, we are going to prove a stronger result; we will see in Theorem 29 that the first order logic given by $[0,1]_\Pi$ coincides with the one given by its one-generated subalgebra. First of all, we notice that every non trivial one-generated subalgebra generates the same first order logic since all these subalgebras are σ-isomorphic. Theorem 29 is stronger than Proposition 5 because (i) all interpretations over a one-generated subalgebra are trivially quasi-witnessed, and (ii) it is well known that concepts can be seen as particular cases of first order formulas.

In (Hájek 1998, Theorem 5.4.30) the author proves that $[0,1]_L$-tautologies coincide with the common $Ł_n$-tautologies for $n \geq 2$, i.e., coincide with the common tautologies of the finite subalgebras of $[0,1]_L$. In (Esteva, Godo, and Noguera 2010) the authors prove that the result is not valid for a logic of a t-norm different from Łukasiewicz. But Hájek's result can be read in another way since L_n are the one-generated subalgebras of $[0,1]_L$ whose generator is a rational number. What we prove in this appendix is that this reading of Hájek's result can be generalized to First Order Product Logic.

In order to prove this result we first prove some lemmas and provide some definitions. Firstly we prove the following lemma that uses only residuation condition, and thus it is also true for any MTL-chain (prelinear residuated chain).

Lemma 25. *In any* Π*-chain the following inequalities hold:*

1. $(x \Leftrightarrow x') * (y \Leftrightarrow y') \leq (x \Rightarrow y) \Leftrightarrow (x' \Rightarrow y')$,
2. $(x \Leftrightarrow x') * (y \Leftrightarrow y') \leq (x * y) \Leftrightarrow (x' * y')$,
3. $\inf_{i\in I}\{x_i \Leftrightarrow y_i\} \leq \inf_{i\in I}\{x_i\} \Leftrightarrow \inf_{i\in I}\{y_i\}$,
4. $\inf_{i\in I}\{x_i \Leftrightarrow y_i\} \leq \sup_{i\in I}\{x_i\} \Leftrightarrow \sup_{i\in I}\{y_i\}$.

Proof. The proofs are easy consequences of residuation property

$$x * y \leq z \qquad \text{iff} \qquad x \leq y \Rightarrow z. \qquad \text{(res)}$$

In particular we point out that $x * (x \Rightarrow y) \leq y$. Next we prove each one of the items.

1. By symmetry it is enough to prove that $(x' \Rightarrow x) * (y \Rightarrow y') \leq (x \Rightarrow y) \Rightarrow (x' \Rightarrow y')$; and this is a consequence of residuation.
2. By symmetry it is enough to prove that $(x \Rightarrow x') * (y \Rightarrow y') \leq (x * y) \Rightarrow (x' * y')$; and this is a consequence of residuation.
3. Since we are considering a chain, we can suppose, without loss of generality, that $\inf_{i\in I}\{y_i\} \leq \inf_{i\in I}\{x_i\}$. Thus, $\inf_{i\in I}\{x_i\} \Leftrightarrow \inf_{i\in I}\{y_i\} = \inf_{i\in I}\{x_i\} \Rightarrow \inf_{i\in I}\{y_i\}$. It is obvious that it is enough to prove that

$$\inf_{i\in I}\{x_i \Rightarrow y_i\} \leq \inf_{i\in I}\{x_i\} \Rightarrow \inf_{i\in I}\{y_i\},$$

and this is an easy consequence of residuation because for every $i \in I$,

$$\inf_{i\in I}\{x_i \Rightarrow y_i\} * \inf_{i\in I}\{x_i\} \leq (x_i \Rightarrow y_i) * x_i \leq y_i.$$

4. Without loss of generality we can assume that $\sup_{i\in I}\{y_i\} \leq \sup_{i\in I}\{x_i\}$. Thus, $\sup_{i\in I}\{x_i\} \Leftrightarrow \sup_{i\in I}\{y_i\} = \sup_{i\in I}\{x_i\} \Rightarrow \sup_{i\in I}\{y_i\}$. It is obvious that it is enough to prove that

$$\inf_{i\in I}\{x_i \Rightarrow y_i\} \leq \sup_{i\in I}\{x_i\} \Rightarrow \sup_{i\in I}\{y_i\}.$$

This is true because if $a = \inf_{i\in I}\{x_i \Rightarrow y_i\}$, then for every $i \in I$,

$$a * x_i \leq y_i;$$

and hence,

$$a * \sup_{i\in I}\{x_i\} = \sup_{i\in I}\{a * x_i\} \leq \sup_{i\in I}\{y_i\}. \qquad \square$$

The proof we give for Theorem 29 is based on a continuity argument, and resembles the one given in (Hájek 1998, Theorem 5.4.30). The main difference is that while Hájek introduces a *distance* between models on the same domain, in this paper we consider a dual notion, which we call *similarity* and denote by S. In the case of Łukasiewicz, since the duality, there is no essential difference between considering a distance or a similarity, but this is not the case for Product Logic, where it is crucial to consider a similarity.

Definition 26 (Similarity). Let Γ be a predicate language with a finite number of predicate symbols $P_1, \ldots, P_n$, and let $\mathbf{M}, \mathbf{M}'$ be two models over $[0,1]_\Pi$ on the same domain M such that r_{P_i} and r'_{P_i} are the interpretations of the predicate symbols in $\mathbf{M}$ and $\mathbf{M}'$ respectively.

1. For each predicate symbol $P \in \Gamma$ with arity $ar(P)$, we define

$$S(r_P, r'_P) := \inf_{a\in M^{ar(P)}}\{r_P(a) \Leftrightarrow r'_P(a)\}$$

$$= \inf_{a\in M^{ar(P)}}\left\{\frac{\min\{r_P(a), r'_P(a)\}}{\max\{r_P(a), r'_P(a)\}}\right\}$$

2. Moreover, we define

$$S(\mathbf{M}, \mathbf{M}') := S(r_{P_1}, r'_{P_1}) * \ldots * S(r_{P_n}, r'_{P_n}).$$

Definition 27. We define the complexity $\tau(\varphi)$ of a formula φ as follows:

1. $\tau(\varphi) = 0$, if φ is atomic or $\perp$,
2. $\tau(\varphi * \psi) = 1 + max\{\tau(\varphi), \tau(\psi)\}$, if $* \in \{\rightarrow, \odot\}$,
3. $\tau(Qx\,\varphi) = \tau(\varphi)$, if $Q \in \{\forall, \exists\}$.

This complexity captures the number of nested propositional connectives in the formula.

Lemma 28. *Assume Γ is a predicate language with n predicate symbols. Let $\mathbf{M}$ and $\mathbf{M}'$ be two first order structures over $[0,1]_\Pi$ on the same domain M, and let φ be a first order formula. Then, for all $\varepsilon \in [0,1)$,*

if $S(\mathbf{M}, \mathbf{M}') > \sqrt[n\cdot 2^{\tau(\varphi)}]{\varepsilon}$, then,
for each evaluation v, $(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon$.

Proof. It is enough to prove that if $\mathbf{M}$ differs from $\mathbf{M}'$ only by the interpretation of one predicate symbol P, then

(C_φ) for all $\varepsilon \in [0,1)$, if $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\varphi)}]{\varepsilon}$, then, for each evaluation v, $(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon$.

We show that this condition (C_φ) holds by induction on the length of the formula φ.

- If φ is either atomic or $\perp$, then it is obvious.
- Let us suppose $\varphi = \psi \star \chi$ with $\star \in \{\odot, \rightarrow\}$, and $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\varphi)}]{\varepsilon}$. Then, $S(\mathbf{M}, \mathbf{M}') > \max\{\sqrt[2^{\tau(\psi)}]{\sqrt{\varepsilon}}, \sqrt[2^{\tau(\chi)}]{\sqrt{\varepsilon}}\}$. Using the inductive hypothesis for $\sqrt{\varepsilon}$, we get that

$$(\|\psi\|_{\mathbf{M},v} \Leftrightarrow \|\psi\|_{\mathbf{M}',v}) \geq \sqrt{\varepsilon},$$

$$(\|\chi\|_{\mathbf{M},v} \Leftrightarrow \|\chi\|_{\mathbf{M}',v}) \geq \sqrt{\varepsilon}.$$

Hence, by the first two items in Lemma 25 we get that

$$(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \sqrt{\varepsilon} * \sqrt{\varepsilon} = \varepsilon.$$

- Let us suppose that $\varphi = Qx\,\psi$, with $Q \in \{\forall, \exists\}$, and $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\varphi)}]{\varepsilon}$. Then, $S(\mathbf{M}, \mathbf{M}') > \sqrt[2^{\tau(\psi)}]{\varepsilon}$. By the inductive hypothesis we get that $(\|\psi\|_{\mathbf{M},v} \Leftrightarrow \|\psi\|_{\mathbf{M}',v}) \geq \varepsilon$ for each evaluation v. Hence,

$$\inf_v\{\|\psi\|_{\mathbf{M},v} \Leftrightarrow \|\psi\|_{\mathbf{M}',v}\} \geq \varepsilon.$$

By the last two items in Lemma 25 it follows that

$$(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon.$$

Hence, the lemma is proved. $\square$

We are now ready to prove the generalization of Proposition 5.

Theorem 29. *A first-order formula φ is a $[0,1]_\Pi$-tautology if and only if it is a tautology in any one-generated subalgebra of $[0,1]_\Pi$.*

Proof. The result is an obvious consequence of the previous lemma. Suppose that φ is not a $[0,1]_\Pi$-tautology, then there is a structure $\mathbf{M}$ and an evaluation v such that $\|\varphi\|_{\mathbf{M},v} < \varepsilon$ for some $\varepsilon < 1$. Take $s \in (0,1)$ such that $s^n > \sqrt[n\cdot 2^{\tau(\varphi)}]{\varepsilon}$, and denote by $\langle s\rangle$ the subalgebra of $[0,1]$ generated by s. For every predicate symbol P, let $r'_P(a)$ be $\min\{t \in \langle s\rangle \mid t \geq r_P(a)\}$. Now we define the structure $\mathbf{M}' = (M, r'_{P_1}, \ldots, r'_{P_n})$ over the algebra $\langle s\rangle$. An easy computation shows that $S(r_P, r'_P) \geq s$ for every predicate symbol P; hence, $S(\mathbf{M}, \mathbf{M}') \geq s^n > \sqrt[n\cdot 2^{\tau(\varphi)}]{\varepsilon}$. By Lemma 28, $(\|\varphi\|_{\mathbf{M},v} \Leftrightarrow \|\varphi\|_{\mathbf{M}',v}) \geq \varepsilon$. This together with the fact that $\|\varphi\|_{\mathbf{M},v} < \varepsilon$ implies that $\|\varphi\|_{\mathbf{M}',v} \neq 1$. This finishes the proof. $\square$

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