On the Interrelationships Among C-Inference Relations Based on Preferred Models for Sets of Default Rules

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Abstract
Various semantics have been developed for knowledge bases consisting of qualitative conditionals representing default rules. Recently, skeptical, weakly skeptical, and credulous inference relations based on c-representations and taking classes of preferred models into account have been proposed. In this paper, we investigate their interrelationships and solve several open problems regarding these interrelationships. In particular, we prove that the preferred models obtained from three different notions of minimality lead to pairwise distinct inference relations, and that none of them is able to exactly capture skeptical c-inference over all c-representations.

1 Introduction
Conditionals like If A, then normally B can be seen as default rules that express a plausible relation between A and B. Various semantics have been developed for knowledge bases consisting of such conditionals, e.g. (Pearl 1990; Benferhat, Dubois, and Prade 1999), or the approach of c-representations (Kern-Isberner 2001; 2004). For the latter, a recent publication (Beierle et al. 2016) proposes various inference modes – skeptical, weakly skeptical, credulous – taking different classes of preferred models into account. In this paper, we show that the preferred model classes induced by the different minimality notions employed in (Beierle et al. 2016) yield pairwise distinct inference relations. We prove that, although they all approximate reasoning over all c-representations, none of them captures this exactly. Furthermore, we argue that weakly skeptical reasoning with respect to the c-representations obtained from the so-called Pareto-minimal impact vectors can allow for more plausible inferences than the other inference variants, while avoiding the disadvantages of liberal credulous c-inference.

After briefly recalling the background of conditional logics and c-inference as far as needed here (Sec. 2 and 3), we elaborate properties of skeptical (Sec. 4), credulous (Sec. 5), and weakly skeptical (Sec. 6) c-inference, before concluding and pointing out further work (Sec. 7).

2 Background
Conditional Logic and OCFs Let L be a propositional language over a finite set Σ of atoms a, b, c, . . . . The formulas of L will be denoted by uppercase Roman letters A, B, C, . . . . For conciseness of notation, we will omit the logical and-connective, writing AB instead of A ∧ B, and overlining formulas will indicate negation, i.e. ̄A means ¬A. Let Ω denote the set of possible worlds over L; Ω will be taken simply as the set of all propositional interpretations over L and can be identified with the set of all complete conjunctions over Σ. For ω ∈ Ω, ω |= A means that the propositional formula A ∈ L holds in the possible world ω.

By introducing a new binary operator |, we obtain the set (L | L) = {(B|A) | A, B ∈ L} of conditionals over L. (B|A) formalizes “if A then (normally) B” and establishes a plausible, probable, possible etc connection between the antecedent A and the consequence B. Here, conditionals are supposed not to be nested, that is, antecedent and consequent of a conditional will be propositional formulas. A conditional (B|A) is an object of a three-valued nature, partitioning the set of worlds Ω in three parts: those worlds satisfying AB, thus verifying the conditional, those worlds satisfying AB, thus falsifying the conditional, and those worlds not fulfilling the premise A and so which the conditional may not be applied to at all. This allows us to associate to (B|A) a generalized indicator function χ(B|A) going back to (de Finetti 1937) (where u stands for unknown or indeterminate):
suitable degree of acceptance is calculated from the degrees associated with $AB$ and $A\overline{B}$.

In this paper, we consider Spohn’s OCFs (Spohn 1988). An OCF is a function $\kappa : \Omega \to \mathbb{N}$ expressing degrees of plausibility of propositional formulas where a higher degree denotes “less plausible” or “more surprising”. At least one world must be regarded as being normal; therefore, $\kappa(\omega) = 0$ for at least one $\omega \in \Omega$. For expressing certain knowledge, the codomain of $\kappa$ can be extended to $\mathbb{N} \cup \{\infty\}$. Each such ranking function can be taken as the representation of a full epistemic state of an agent. Each such $\kappa$ uniquely extends to a function (also denoted by $\kappa$) mapping sentences to $\mathbb{N} \cup \{\infty\}$ by:

$$\kappa(A) = \begin{cases} \min \{\kappa(\omega) \mid \omega \models A\} & \text{if } A \text{ is satisfiable} \\ \infty & \text{otherwise} \end{cases} \quad (2)$$

An OCF $\kappa$ accepts a conditional $\langle B \mid A \rangle$ (denoted by $\kappa \models \langle B \mid A \rangle$) if and only if the verification of the conditional is less surprising than its falsification, i.e., if, and only if $\kappa(AB) < \kappa(A\overline{B})$. This can also be understood as a non-monotonic inference relation between the premise $A$ and the conclusion $B$: We say that $A$ $\kappa$-entails $B$ (written $A \models_{\kappa} B$) if and only if $\kappa(\langle B \mid A \rangle)$, formally

$$A \models_{\kappa} B \iff \kappa(\langle B \mid A \rangle) \leq \kappa(\langle A \overline{B} \mid A \rangle). \quad (3)$$

A set $\mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L})$ of conditionals is called a knowledge base if it does not contain any self-fulfilling or contradictory conditional. An OCF $\kappa$ accepts a knowledge base $\mathcal{R}$ iff $\kappa$ accepts all conditionals in $\mathcal{R}$, and $\mathcal{R}$ is consistent iff an OCF accepting $\mathcal{R}$ exists (Pearl 1990).

**C-Representations** Different ways of determining a ranking function for a knowledge base $\mathcal{R}$ have been proposed, e.g., the well-known system $Z$ (Goldszmidt and Pearl 1996). Here, we will use c-representations that assign an individual impact $\eta_i$ to each conditional $\langle B_i \mid A_i \rangle$ and generate the world ranks as a sum of impacts of falsified conditionals:

**Definition 1** (c-representation) (Kern-Isberner 2001; 2004). A c-representation of a knowledge base $\mathcal{R} = \{\langle B_1 \mid A_1 \rangle, \ldots, \langle B_n \mid A_n \rangle\}$ is an OCF $\kappa$ constructed from non-negative integer impacts $\eta_i \in \mathbb{N}$ assigned to each conditional $\langle B_i \mid A_i \rangle$ such that $\kappa$ accepts $\mathcal{R}$ and is given by:

$$\kappa(\omega) = \sum_{1 \leq i \leq n} \eta_i \quad (4)$$

C-representations can conveniently be specified using a constraint satisfaction problem (for detailed explanations, see (Kern-Isberner 2001; 2004; Beierle, Eichhorn, and Kern-Isberner 2016)).

**Definition 2** ($CR(\mathcal{R})$) (Beierle, Eichhorn, and Kern-Isberner 2016). Let $\mathcal{R} = \{\langle B_1 \mid A_1 \rangle, \ldots, \langle B_n \mid A_n \rangle\}$. The constraint satisfaction problem for c-representations of $\mathcal{R}$, denoted by $CR(\mathcal{R})$, is given by the conjunction of the constraints, for all $i \in \{1, \ldots, n\}$:

$$\eta_i \geq 0 \quad \min_{\omega \models A_i B_i} \sum_{j \neq i} \eta_j - \min_{\omega \models A_j \overline{B}_j} \sum_{j \neq i} \eta_j \quad (5)$$

A solution of $CR(\mathcal{R})$ is an $n$-tuple $(\eta_1, \ldots, \eta_n)$ of natural numbers. For a constraint satisfaction problem $CSP$, the set of solutions is denoted by $Sol(CSP)$. Thus, with $Sol(CR(\mathcal{R}))$ we denote the set of all solutions of $CR(\mathcal{R})$. For any $\eta \in Sol(CR(\mathcal{R}))$ and $\kappa$ as in Equation (4), $\kappa$ is the OCF induced by $\eta$ and is denoted by $\kappa(\eta)$.

**Proposition 3** (Correctness and Completeness of $CR(\mathcal{R})$) (Beierle, Eichhorn, and Kern-Isberner 2016). Let $\mathcal{R} = \{\langle B_1 \mid A_1 \rangle, \ldots, \langle B_n \mid A_n \rangle\}$ be a knowledge base. For every $\eta \in Sol(CR(\mathcal{R}))$, $\kappa(\eta)$ is a c-representation with $\kappa(\eta) \models \mathcal{R}$, and for every c-representation $\kappa$ with $\kappa \models \mathcal{R}$, there is a $\eta \in Sol(CR(\mathcal{R}))$ such that $\kappa = \kappa(\eta)$.

### 3 Inference With C-Representations

Whereas Equation (3) defines an inference relation $\models_{\kappa}$ based on a single OCF $\kappa$, skeptical c-inference takes all c-representations of a given knowledge base $\mathcal{R}$ into account.

**Definition 4** (skeptical c-inference, $\models_{s\kappa}^{CR}$) (Beierle et al. 2016). Let $\mathcal{R}$ be a knowledge base and let $A$, $B$ be formulas. $B$ is a skeptical c-inference from $A$ in the context of $\mathcal{R}$, denoted by $A \models_{s\kappa}^{CR} B$, iff there is a c-representation $\kappa$ for $\mathcal{R}$ such that $A \models_{\kappa} B$ holds.

Any single c-representation accepting $\mathcal{R}$ can be used for defining an inference relation inductively completing $\mathcal{R}$ that can be used as a prototype while exhibiting desirable properties (Thorn et al. 2015); this observation provides practical relevance for the following definition.

**Definition 5** (credulous c-inference, $\models_{c\kappa}^{CR}$) (Beierle et al. 2016). Let $\mathcal{R}$ be a knowledge base and let $A$, $B$ be formulas. $B$ is a credulous c-inference from $A$ in the context of $\mathcal{R}$, denoted by $A \models_{c\kappa}^{CR} B$, iff there is a c-representation $\kappa$ for $\mathcal{R}$ such that $A \models_{\kappa} B$ holds.

Credulous c-inference is a liberal extension of skeptical c-inference since $A \models_{s\kappa}^{CR} B$ implies $A \models_{c\kappa}^{CR} B$ for any consistent knowledge base $\mathcal{R}$ and any formulas $A$, $B$. However, credulous c-inference has the disadvantage that we might have both, $A \models_{c\kappa}^{CR} B$ and $A \models_{c\kappa}^{CR} \overline{B}$. Lying between skeptical and credulous inference, weakly skeptical c-inference is strictly more liberal than skeptical inference, but less permissive than credulous inference.

**Definition 6** (weakly skeptical c-inference, $\models_{ws\kappa}^{CR}$) (Beierle et al. 2016). Let $\mathcal{R}$ be a knowledge base and let $A$, $B$ be formulas. $B$ is a weakly skeptical c-inference from $A$ in the context of $\mathcal{R}$, denoted by $A \models_{ws\kappa}^{CR} B$, iff there is a c-representation $\kappa$ for $\mathcal{R}$ such that $A \models_{\kappa} B$ holds and there is no c-representation $\kappa'$ for $\mathcal{R}$ such that $A \models_{\kappa'} \overline{B}$.

The following example shows that weakly skeptical c-inference allows for some desirable inferences that are not possible under skeptical c-inference.

**Example 7.** Let $\mathcal{R}_{fb} = \{r_1, r_2, r_3\}$ be given by:

$$r_1 : (f \mid b) \quad \text{birds fly}$$
$$r_2 : (a \mid b) \quad \text{birds are animals}$$
$$r_3 : (a \mid f) \quad \text{flying birds are animals}$$

Consider a bird that lost its ability to fly ($b$); we would expect that it is still considered an animal ($a$). Yet it holds
that \( b \vdash \eta \sim_{R,sk} a \). On the other hand, both \( b \vdash \eta \sim_{R,cw} a \) and \( b \vdash \eta \sim_{R,ws} a \) hold. Therefore, from Def. 6 we get \( b \vdash \eta \sim_{R,sk} a \).

Weakly skeptical inference lies indeed strictly between skeptical and credulous inference.

**Proposition 8 ((Beierle et al. 2016)).** For every consistent knowledge base \( R \):

\[
\vdash_{sk} R \subseteq \vdash_{ws} R \subseteq \vdash_{cw} R
\]

(7)

From the point of view of minimal specificity, c-representations yielding minimal degrees of implausibility are most interesting. Different orderings on \( \text{Sol}(\text{CR}(R)) \) leading to different minimality notions can be used.

**Definition 9 (sum-, cw-, ind-minimal (Beierle et al. 2016)).** Let \( R \) be a knowledge base and \( \eta, \eta' \in \text{Sol}(\text{CR}(R)) \).

\[
(\eta_1, \ldots, \eta_n) \preceq_{+} (\eta'_1, \ldots, \eta'_n) \quad \text{iff} \quad \sum_{1 \leq i \leq n} \eta_i \leq \sum_{1 \leq i \leq n} \eta'_i.
\]

(8)

\( \eta \) is sum-minimal if \( \eta \preceq_{+} \eta' \) for all \( \eta' \in \text{Sol}(\text{CR}(R)) \). We write \( \eta \prec_{+} \eta' \) if \( \eta \preceq_{+} \eta' \) and \( \eta \neq \eta' \).

\[
(\eta_1, \ldots, \eta_n) \preceq_{cw} (\eta'_1, \ldots, \eta'_n) \quad \text{iff} \quad \eta_i \leq \eta'_i \text{ for all } i \in \{1, \ldots, n\}.
\]

(9)

\( \eta \) is cw-minimal (or Pareto-minimal) if there is no vector \( \eta' \in \text{Sol}(\text{CR}(R)) \) such that \( \eta \preceq_{cw} \eta' \) and \( \eta \neq \eta' \).

\[
(\eta_1, \ldots, \eta_n) \preceq_{O} (\eta'_1, \ldots, \eta'_n) \quad \text{iff} \quad \kappa_\eta(\omega) \preceq \kappa_{\eta'}(\omega) \text{ for all } \omega \in \Omega.
\]

(10)

\( \eta \) is ind-minimal if there is no vector \( \eta' \in \text{Sol}(\text{CR}(R)) \) such that \( \eta \preceq_{O} \eta' \) and \( \eta \neq \eta' \).

Thus, while sum-minimal and cw-minimal are defined by just taking the components of the solution vectors \( \eta \) into account, ind-minimality refers to the ranking function induced by a solution vector.

**Example 10.** Consider \( R_{bf} \) from Ex. 7. From (6), we get

\[
\eta_1 > 0, \quad \eta_2 > 0 - \min\{\eta_1, \eta_3\}, \quad \eta_3 > 0 - \eta_2
\]

and since \( \eta_i \geq 0 \) according to (5), the two vectors \( \bar{\eta}_1 = (1, 1, 0) \), \( \bar{\eta}_2 = (1, 0, 1) \) are two solutions of \( \text{CR}(R_{bf}) \) that are both sum-minimal and cw-minimal in \( \text{Sol}(R_{bf}) \). Only \( \bar{\eta}_3 = (2) \) is ind-minimal because \( \kappa_{\bar{\eta}_1}(\bar{\eta}_1) = 1 < 2 = \kappa_{\bar{\eta}_2}(\bar{\eta}_2) \).

Each of the ordering relations \( \preceq_{O} \) with \( \bullet \in \{+, \text{cw}, O\} \) induces a set of solutions of \( \text{CR}(R) \), denoted by

\[
\text{Sol}^{\min}_{\bullet}(\text{CR}(R)) = \{ \eta \mid \eta \in \text{Sol}(\text{CR}(R)) \text{ and } \eta \text{ is } \bullet \text{-minimal}\}
\]

that are minimal with respect to \( \preceq_{O} \). Viewing these minimal models as preferred models, skeptical, credulous, and weakly skeptical inference versions are obtained from the definitions of \( \vdash_{sk} R \), \( \vdash_{cw} R \), and \( \vdash_{ws} R \) by replacing the set \( \text{Sol}(\text{CR}(R)) \) by the respective set of minimal solutions.

**Definition 11 (min-inference).** Let \( R \) be a knowledge base, let \( A, B \) be formulas, and let \( \bullet \in \{+, \text{cw}, O\} \).

1. \( B \) is a skeptical \( \bullet \)-min-inference from \( A \) in the context of \( R \), denoted by \( A \vdash_{sk} B \), if and only if \( A \vdash B \) holds for all \( \kappa \in \text{Sol}^{\min}_{\bullet}(\text{CR}(R)) \).

2. \( B \) is a credulous \( \bullet \)-min-inference from \( A \) in the context of \( R \), denoted by \( A \vdash_{cw} B \), if and only if there is a \( \kappa \in \text{Sol}^{\min}_{\bullet}(\text{CR}(R)) \) such that \( A \vdash \kappa B \).

3. \( B \) is a weakly skeptical \( \bullet \)-min-inference from \( A \) in the context of \( R \), denoted by \( A \vdash_{O} B \), if and only if there is no \( \kappa' \in \text{Sol}^{\min}_{\bullet}(\text{CR}(R)) \) such that \( A \vdash \kappa' \).

The inclusions given in Prop. 8 carry over to all three kinds of min-inference.

**Proposition 12 ((Beierle et al. 2016)).** For every consistent knowledge base \( R \) and \( \bullet \in \{+, \text{cw}, O\} \) we have:

\[
\vdash_{sk} \bullet \subseteq \vdash_{ws} \bullet \subseteq \vdash_{cw} \bullet
\]

(11)

Propositions 8 and 12 state only some of the interrelationships among the different inference modes based on c-representations. In the following, we will systematically investigate the interrelationships among the different c-inference relations, thereby solving several of the open problems given in (Beierle et al. 2016).

### 4 Skeptical C-Inference

Proposition 12 clarifies the interrelationships of the respective c-inference relations across the different inference modes skeptical, weakly skeptical, and credulous as being strict subset relations. So far, less has been known about the interrelationships among the different classes of preferred models within each of the inference modes. For skeptical inference with respect to a consistent knowledge base \( R \), the subset relations

\[
\vdash_{sk} R \subseteq \vdash_{ws} R \subseteq \vdash_{cw} R
\]

(12)

\[
\vdash_{sk,0} R \subseteq \vdash_{sk} R \subseteq \vdash_{cw} R
\]

(13)

for \( \bullet \in \{+, \text{cw}, O\} \) and \( \circ \in \{+, \text{cw}, O\} \) have been established in (Beierle et al. 2016). In this section, the inverse relations of (12) and (13) will be investigated.

**Skeptical cw-min vs. sum-min Inference** Of the relations (12) and (13), only (13) for \( \circ = + \) was shown to be strict in (Beierle et al. 2016), i.e., there is a knowledge base \( R \) such that:

\[
\vdash_{cw} R \not\subseteq \vdash_{sk} R
\]

(14)

**Skeptical cw-min vs. ind-min Inference** However, it has been an open question whether the opposite direction of (13) also holds for \( \circ = O \), i.e., whether every skeptical cw-inference is also a ind-min inference. The following proposition shows that this is not the case.

**Proposition 13.** There is a knowledge base \( R \) such that:

\[
\vdash_{sk,0} R \not\subseteq \vdash_{cw} R
\]

(15)
Table 1: Verification (v), falsification (f), impacts (ηi), and solution vectors \( \vec{\eta} \), \( \vec{\eta} \), \( \vec{\eta} \), and their induced OCFs for knowledge base \( \mathcal{R} \) used in the proof of Proposition 13.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \eta_1 )</th>
<th>( \eta_2 )</th>
<th>( \eta_3 )</th>
<th>( \eta_4 )</th>
<th>impact on ( \omega )</th>
<th>( \kappa_{\vec{\eta}^{1}} ) (( \omega ))</th>
<th>( \kappa_{\vec{\eta}^{2}} ) (( \omega ))</th>
<th>( \kappa_{\vec{\eta}^{3}} ) (( \omega ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{\eta} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>cw-, sum-, ind-min.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \vec{\eta} )</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>cw-min.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \vec{\eta} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proper. Consider \( \mathcal{R} = \{ (b(a), (b)c), (c)d, (d)\} \) over the atoms \( \Sigma = \{ a, b, c, d \} \). Table 1 shows the verification and falsification behavior of all worlds with respect to the four conditionals in \( \mathcal{R} \) and the induced impacts for each world. For the logical interdependencies in \( \mathcal{R} \), \( CR(\mathcal{R}) \) is given by the conjunction of the following constraints:

\[
\eta_i \geq 0 \quad \text{for } 1 \leq i \leq 4 \quad (16)
\]
\[
\eta_i > 0 \quad (17)
\]
\[
\eta_i > 0 \quad (18)
\]
\[
\eta_i > 0 \quad (19)
\]
\[
\eta_i > \eta_i - \min \{ \eta_1, \eta_2 \} \quad (20)
\]

Both \( \vec{\eta} = (\eta_1, ..., \eta_4) = (1, 1, 1, 1) \) and \( \vec{\eta} = (\eta_2, ..., \eta_4) = (2, 2, 1, 0) \) are solutions of \( CR(\mathcal{R}) \) as can easily be checked by substituting the respective values in (16) - (20). Their induced OCFs are also given in Table 1.

We will now prove that \( \{ \vec{\eta}^{1}, \vec{\eta}^{2} \} \) is the set of all cw-minimal solutions. To see that, assume that there was \( \vec{\eta} = (\eta_1', \eta_2', \eta_3', \eta_4') \in Sol(\mathcal{R}) \) with \( \vec{\eta}^{1} \neq cw \vec{\eta} \) and \( \vec{\eta}^{2} \neq cw \vec{\eta} \). It cannot be the case that \( \eta_i' < \eta_i \) for \( i \in \{ 1, 2, 3 \} \) because of (17), (18), and (19), leaving \( \eta_i' < \eta_i \) as the only possibility. From \( \eta_i' < \eta_i \) and (16), we get

\[
0 \leq \eta_i' = 0 < \eta_i, \quad (21)
\]

From (20) we then get \( 0 > \eta_i' - \min \{ \eta_1', \eta_2' \} \) and therefore \( \eta_i < \min \{ \eta_1', \eta_2' \} \). This leads to \( 0 < \eta_i' < \min \{ \eta_1', \eta_2' \} \) due to (19). In particular, this implies \( \min \{ \eta_1, \eta_2 \} > 1 \) and consequently, both

\[
\eta_1' > 1 \quad (22)
\]
\[
\eta_2' > 1 \quad (23)
\]

must hold. Because of \( \eta_2' \neq cw \eta' \), it is required that for one \( i \in \{ 1, 2, 3, 4 \} \) it is the case that \( \eta_i' < \eta_i \). Therefore, one of the following four conditions

\[
0 < \eta_i' < \eta_i \quad (24)
\]
\[
0 < \eta_i' < \eta_i \quad (25)
\]
\[
0 < \eta_i' < \eta_i \quad (26)
\]
\[
0 = \eta_i' < \eta_i \quad (27)
\]

must be true. Conditions (26) and (27) are, on their own, not satisfiable and (24) and (25) contradict (22) and (23), respectively. This contradicts the assumption and shows that \( \vec{\eta}^{1} \) and \( \vec{\eta}^{2} \) are the only cw-minimal solutions.

We observe that \( \kappa_{\vec{\eta}}(\omega) \leq \kappa_{\vec{\eta}}(\omega) \) for all \( \omega \), and that, e.g., \( \kappa_{\vec{\eta}}(ab\eta d) = 2 < 3 = \kappa_{\vec{\eta}}(ab\eta d) \) (cf. Table 1). Thus, since every ind-minimal solution is also cw-minimal, we conclude that \( \vec{\eta}^{1} \) is the only ind-minimal solution of \( Sol(CR(\mathcal{R})) \).

Now consider the conditional (\( \tilde{\eta} \)). Since

\[
\kappa_{\vec{\eta}}(\tilde{\eta}) = 1 < 2 = \kappa_{\vec{\eta}}(\tilde{\eta})
\]

(\( \tilde{\eta} \)) is accepted by the c-representation induced by the only ind-minimal solution \( \vec{\eta}^{1} \). On the other hand, since

\[
\kappa_{\vec{\eta}}(\tilde{\eta}) = 2 = \kappa_{\vec{\eta}}(\tilde{\eta})
\]

the conditional is rejected by the c-representation induced by the cw-minimal c-representation \( \vec{\eta}^{2} \). Therfore, we have

\[
\tilde{\eta} \models sk R \tilde{\eta} \quad \text{and} \quad \tilde{\eta} \models sk, cw R \tilde{\eta}
\]

which completes the proof.

\[
\square
\]

Note that for illustration, Table 1 also displays a non-minimal impact vector \( \vec{\eta}^{3} \) and its induced OCF.

**Relationship Between Skeptical cw-min Inference and Skeptical Inference Over All C-representation**

Another open problem stated in (Beierle et al. 2016) is the question whether skeptical c-inference can be modeled by skeptical cw-min inference. Since skeptical c-inference is defined with respect to the set \( Sol(CR(\mathcal{R})) \) and since the set of cw-minimal solutions is a subset thereof, we trivially have

\[
\neg sk R \subseteq \neg sk, cw R
\]

for every knowledge base \( \mathcal{R} \). The next proposition shows that the inverse direction of (28) does not hold.

**Proposition 14.** There is a knowledge base \( \mathcal{R} \) such that:

\[
\neg sk, cw R \not\subseteq \neg sk R
\]

**Proof.** Consider \( \mathcal{R} = \{ (b(a), (b)c), (a)b, c) \} \) over the atoms \( \Sigma = \{ a, b, c \} \). Table 2 shows the verification and falsification behavior of all worlds with respect to the four conditionals in \( \mathcal{R} \) and the induced impacts for each world.
Table 2: Verification (v), falsification (f), impacts (η), and
the solution vectors ̄η1, ̄η2 and their induced OCFs for
knowledge base R used in the proof of Proposition 14.

| ω | \(R_1\): (b|a) | \(R_2\): (b|ac) | \(R_3\): (π|β) | \(R_4\): (π|bc) | impact on ω | \(κ_1\) | \(κ_2\) |
|---|---|---|---|---|---|---|---|
| abc | v | v | f | η₄ | 1 | 1 |  |
| abç | v | v | f | η₃ | 1 | 3 |  |
| aöc | f | f | | η₁ + η₂ | 2 | 2 |  |
| aðc | f | f | | η₁ | 2 | 2 |  |
| ñbc | -- | -- | v | 0 | 0 | 0 |  |
| ñcd | -- | -- | η₁ | 0 | 0 | 0 |  |
| ñbc | -- | -- | | η₁ | 0 | 0 | 0 |  |

The impact vector ̄η₁ = (η₁,...,η₄) = (2,0,1,1) is a solution of \(CR(\mathcal{R})\) as can easily be checked by substituting the values in (30) - (34). The induced OCF \(κ_1\) is given in the seventh column of Table 2.

In the following, we will prove that ̄η₁ is the only cw-minimal solution of \(CR(\mathcal{R})\). In order to show this, let us assume that there was ̄η₂ ∈ \(Sol(CR(\mathcal{R}))\) that is also cw-minimal, but different from ̄η₁. Then there must be a non-empty set \(I\) of indices, \(I \subseteq \{1, 2, 3, 4\}\) for which it is the case that \(η₃^I < η₄^I\) for all \(i \in I\). First we note that 2 \(\notin I\) because \(η₂^I < η₄^I\) violates (30). Furthermore, 3 \(\notin I\) because \(η₃^I < η₄^I\) violates (33) or (34); and 4 \(\notin I\) because \(η₄^I < η₄^I\) violates (34). Finally, \(η₁^I < η₄^I\) would imply \(η₁^I < η₄^I\) which together with (31) implies \(1 > min\{η₃^I, η₄^I\}\), and hence \(I \subseteq \emptyset\). These observations imply \(I = \emptyset\), contradicting the assumption that ̄η₂ is cw-minimal and establishing that ̄η₁ is the only cw-minimal element of \(Sol(CR(\mathcal{R}))\).

Now consider the conditional (b|a|c). For the single cw-minimal c-representation ̄η₁ we have

\[κ_1(a|b) = 1 < 2 = κ_1(a|b)\]

and therefore, \(a|b| \not\vdash_{k,cw} b\).

The impact vector ̄η₂ = (2,0,3,1) is also a solution of \(CR(\mathcal{R})\) given in (30) - (34). The induced OCF \(κ_2\) is given in the last column of Table 2. For \(κ_2\) we have

\[κ_2(a|b) = 3 < 2 = κ_2(a|b)\]

and therefore, \(a|b| \not\vdash_{k,cw} b\).

\[a|b| \not\vdash_{k,cw} b\]  

which together with (35) completes the proof. □

5 Credulous C-Inference

For credulous inference with respect to a consistent knowledge base \(\mathcal{R}\), the subset relations

\[\vdash_{\mathcal{R}}^c \subseteq \vdash_{\mathcal{R}}\]  

\[\vdash_{\mathcal{R}} \subseteq \vdash_{\mathcal{R}}^c\]

for for \(\bullet \in \{+, c, O\}\) and \(\circ \in \{+, O\}\) have already been given in (Beierle et al. 2016). In the following, the inverse relations of (37) and (38) will be investigated.

Credulous cw-min vs. sum-min and ind-min Inference

The next proposition shows that not every credulous cw-minimal inference is also a sum- or ind-minimal inference.

Proposition 15. There is a knowledge base \(\mathcal{R}\) such that:

\[\vdash_{\mathcal{R}}^{cw} \not\subseteq \vdash_{\mathcal{R}}^+\]  

\[\vdash_{\mathcal{R}}^{cw} \not\subseteq \vdash_{\mathcal{R}}^O\]

Proof. Let \(\mathcal{R}\) be the knowledge base used in the proof of Prop. 13 (see Table 1). For the conditional (c|a|b|d) we have

\[κ_1(a|b|d) = 2 < 2 = κ_1(a|b|d)\]

and since ̄η₁ is the only sum- and the only ind-minimal solution of \(CR(\mathcal{R})\) we get

\[a|b|d \not\vdash_{\mathcal{R}}^O c\]  

But for the cw-minimal ̄η₂ we have

\[κ_2(a|b|d) = 2 < 3 = κ_2(a|b|d)\]

and therefore

\[a|b|d \vdash_{\mathcal{R}}^{cw} c\]

which together with (41) completes the proof. □

Credulous cw-min Inference vs. Credulous Inference Over All C-Representations

Finally, we show that there are strictly less credulous cw-min-inferences than inferences over all c-representations.

Proposition 16. There is a knowledge base \(\mathcal{R}\) such that:

\[\vdash_{\mathcal{R}}^c \not\subseteq \vdash_{\mathcal{R}}^{cw}\]

Proof. Let \(\mathcal{R}\) be as in the proof of Proposition 14 and consider the conditional (b|a|c). For ̄η₂ (cf. Table 2) we have

\[κ_2(a|b|c) = 2 ≥ 3 = κ_2(a|b|c)\]

and therefore

\[a|b| \not\vdash_{\mathcal{R}}^{cw} b\]  

For ̄η₁, the only cw-minimal solution of \(CR(\mathcal{R})\) (cf. the proof of Proposition 14) we have

\[κ_1(a|b) = 2 < 1 = κ_1(a|b)\]

and therefore

\[a|b| \not\vdash_{\mathcal{R}}^{cw} b\]

which together with (44) completes the proof. □
6 Weakly Skeptical C-Inference

While the subset relationships in Section 4 and 5 reflect the fact that over a subset of models skeptical inference yields a stricter and credulous inference yields a more tolerant inference relation, the interrelationships are not governed by this principle in the case of weakly skeptical inference. The following Proposition shows that there are situations where weakly skeptical inference over a subset of models allows for more plausible inferences than all other variants.

Proposition 17. There is a knowledge base $\mathcal{R}$ such that:

\[ \models_{\mathcal{R}}^\text{ws, cw} \in \models_{\mathcal{R}}\quad \text{(46)} \]

\[ \models_{\mathcal{R}}^\text{ws, cw} \in \models_{\mathcal{R}}^+ \quad \text{(47)} \]

\[ \models_{\mathcal{R}}^\text{ws, cw} \nsubseteq \models_{\mathcal{R}}^\text{ws, O} \quad \text{(48)} \]

Proof. Consider $\mathcal{R} = \{(b|a), (b|c), (c|d), (\overline{c}|ad), (\overline{c}|d)\}$ used in the proofs of Proposition 13 and 15. Table 1 shows all minimal solutions ($\bar{\eta}_1$ and $\bar{\eta}_2$) as well as the solution $\bar{\eta}_3 = (1, 1, 1, 2)$ to $CR(\mathcal{R})$ that is not minimal with respect to any of the minimality criteria. From the proof of Proposition 15 we already know that for the conditional $(c|\overline{a}bd)$

\[ a\overline{bd} \models_{\mathcal{R}}^\text{cr, cw} c. \quad \text{(49)} \]

Since we have both

\[ \kappa_{\bar{\eta}_1}(a\overline{bcd}) = 2 + 2 = \kappa_{\bar{\eta}_2}(a\overline{bcd}) \quad \text{and} \]

\[ \kappa_{\bar{\eta}_3}(a\overline{bcd}) = 3 + 2 = \kappa_{\bar{\eta}_2}(a\overline{bcd}) \]

we know that $a\overline{bd} \models_{\mathcal{R}}^\text{ws, cw} \overline{c}$ which together with (49) shows that

\[ a\overline{bd} \models_{\mathcal{R}}^\text{ws, cw} c. \quad \text{(50)} \]

For proving (46), we show that the inference in (50) does not hold for $\models^\text{ws, R}$, since $\models^\text{cr, R}$ infers both $c$ and $\overline{c}$ from $a\overline{bd}$. In Table 1 we can see that $\kappa_{\bar{\eta}_1}(a\overline{bcd}) = 2 < 3 = \kappa_{\bar{\eta}_2}(a\overline{bcd})$ showing that $a\overline{bd} \models_{\mathcal{R}}^\text{cr} \overline{c}$ which means that $a\overline{bd} \models_{\mathcal{R}}^\text{ws} c$ which together with (50) shows (46). For proving (47) and (48), we show that the inference in (50) does not hold for $\models^\text{ws, R}^+$ or $\models^\text{ws, O}$, since neither $\models^\text{cr, R}$ nor $\models^\text{cr, O}$ infer $c$ from $a\overline{bd}$. Since $\bar{\eta}_1$ is the only sum-minimal and the only ind-minimal solution to $CR(\mathcal{R})$, $\models^\text{ws, R}^+$ and $\models^\text{ws, O}$ coincide. We already know from (41) in the proof of Proposition 15 that $a\overline{bd} \models_{\mathcal{R}}^\text{cr, R} c$ so we also have

\[ a\overline{bd} \models_{\mathcal{R}}^\text{ws, R}^+ c \quad \text{and} \quad a\overline{bd} \models_{\mathcal{R}}^\text{ws, O} c \quad \text{(51)} \]

which together with (50) shows (47) and (48), respectively.

\[ \square \]

This shows that even with the same knowledge base and query, weakly skeptical inference both over a subset or over a superset of models can lead to a more tolerant inference relation. When comparing to inference over all c-representations, inference over cw-minimal solutions allows for more weakly skeptical inferences, because it does not take models into account that allow for contradicting consequences. When comparing to inference over sum- or ind-minimal c-representations, inference over cw-minimal c-representations allows for more inferences, because it takes more models into account that allow for desirable inferences. The exact relationships still need to be investigated.

7 Conclusions and Future Work

We elaborated the interrelationships among c-inference relations induced by various inference modes and differing model classes, and proved, e.g., that skeptical inference over all c-representations can not be captured exactly by any of the other inference variants. Our current work includes extending the investigation of (Beierle et al. 2016) which general axioms proposed for conditional reasoning (e.g. in (Kraus, Lehmann, and Magidor 1990; Lukasiewicz 2005)) are satisfied by the individual c-inference variants and how these inference relations compare to other approaches, e.g. (Benferhat, Dubois, and Prade 2002).

References


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