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Propositional Belief Merging with OWA Operators

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Abstract

An Ordered Weighted Averaging (OWA) operator provides a parameterized family of aggregation operators which include many of the well-known operators such as the maximum, the minimum and the mean. We introduce OWA operators as propositional belief merging operators and investigate their logical properties, as well as their relation with IC and pre-IC merging operators.

Introduction

The OWA operators were originally introduced in (Yager 1988) to provide means for aggregating information. They have proved to be a useful family of aggregation operators which have a fundamental aspect of assigning weights to the values being aggregated.

On the other hand, the area of propositional belief merging studies the fusion of independent and equally reliable sources of information expressed in propositional logic, and considers some aspects of rationality. In this paper, we will focus on model-based operators, where we will be using the notions of IC merging operators. An IC merging operator satisfies nine basic IC logical postulates (**(IC0)**-**(IC8)**). Besides, there are two sub-classes of belief merging operators: utilitarian and egalitarian operators (Konieczny and Pino-Pérez 1999; 2011; Everaere, Konieczny, and Marquis 2014).

When regarding egalitarian operators, it is natural to consider merging operators which tries to achieve a *fair* result. In (Everaere, Konieczny, and Marquis 2014), two egalitarian conditions coming from social choice theory were translated into the propositional belief merging framework: Hammond equity **(HE)** (Hammond 1976) and Pigou-Dalton condition **(PD)** (Dalton 1920). Besides, two new families of belief merging operators based on the median and on a cumulative sum were introduced. A general family of belief merging operators called pre-IC merging operators was defined, by weakening two IC logical postulates (**(IC5)** and **(IC6)**).

One of the aims of this paper is to continue this investigation on egalitarian operators, by introducing OWA merging operators. As our main contributions, we will define OWA merging operators and show their logical properties.

As the operators defined in (Everaere, Konieczny, and Marquis 2014), OWA merging operators we will not satisfy all the usual IC logical postulates. We will show what conditions need to be achieved for an OWA merging operator to satisfy some missing IC logical postulates. Depending on the chosen weights, OWA merging operators can be in the family of IC or pre-IC merging operators.

Belief Merging with OWA Operators

OWA operators (Yager 1988) are a parameterized family of aggregation operators which include many well-known operators such as the maximum, minimum and the simple average (Yager and Kacprzyk 1997).

Definition 1 (OWA Operator) *(Yager 1988) An OWA operator is a mapping* $f_W : \mathbb{R}^n \rightarrow \mathbb{R}$, *such that* $W = [w_1, w_2, \ldots, w_n]$ *is a vector of weights, (1)* $w_i \in [0, 1]$ *and (2)* $\sum_i w_i = 1$. *Furthermore* $f_W(a_1, \ldots, a_n) = \sum_j w_j b_j$, *where* b_j *is the* j*th largest of the* a_i *in* $(a_1, \ldots, a_n)$.

OWA operators are distinguished by their vector of weights. In (Yager 1988) it was pointed out three important cases of vectors: $W^* = [1, 0, \ldots, 0]$; $W_* = [0, \ldots, 0, 1]$; and $W_A = \left[\frac{1}{n}, \ldots, \frac{1}{n}\right]$. W^* gives weight only to the highest value of a vector (whilst W_* gives it to the lowest value) and the rest of the values have no associated weight. W_A associates an equal weight to all values in a vector. It can easily be seen that $f_{W^*}(a_1, \ldots, a_n) = \max_i(a_i)$; $f_{W_*}(a_1, \ldots, a_n) = \min_i(a_i)$; and $f_{W_A}(a_1, \ldots, a_n) = \frac{1}{n}\sum_i a_i$. An OWA Merging Operator may be defined directly in the following way:

Definition 2 (OWA Merging Operators) *Let* d *be a distance measure and* $E = \{K_1, \ldots, K_n\}$ *a belief set. For each outcome* ω, *we consider the vector* $L_E^\omega = (l_1^\omega, \ldots, l_n^\omega)$ *where* $l_i^\omega = d(\omega, K_{\sigma(i)})$ *is the distance between* $K_{\sigma(i)}$ *and* ω, *and* σ *is the permutation of* $\{1, \ldots, n\}$ *such that* $l_i^\omega \geq l_{i+1}^\omega$ *for every* $1 \leq i < n$. *Then we define the vector* $W = [w_1, \ldots, w_n]$, *where* $w_i \in [0, 1]$ *and* $\sum_i w_i = 1$. *Let* $d(W, L_E^\omega) = \sum_{i=1}^{n} w_i l_i^\omega$. *Then we have the following* pre-order: $\omega_i \leq_E^{d,W} \omega_j$ iff $d(W, L_E^{\omega_i}) \leq d(W, L_E^{\omega_j})$. The OWA merging operator $\Delta_\mu^{d,W}$ is defined by $\Delta_\mu^{d,W}(E) = min(mod(\mu), \leq_E^{d,W})$.*

The idea for an OWA merging operator is to give the possibility of allowing different priorities for the information in a group. Different from max and $leximax$, which give priority to the worst case, an OWA is flexible enough to give more or less priority for any position in a group, and consequently dealing with different degrees of priorities.

Theorem 1 *$\Delta_\mu^{d,W}$ satisfies **(IC0)**, **(IC1)**, **(IC3)**, **(IC4)**, **(IC5b)**, **(IC7)** and **(IC8)**.*

In general, an OWA merging operator is not an IC or pre-IC merging operator, since it does not satisfy **(IC2)**, **(IC5)**, **(IC6)** or **(IC6b)**. However, it is possible to state some conditions which validates some of these logical properties.

Theorem 2 *$\Delta_\mu^{d,W}$ satisfies **(IC6b)** if and only if $w_i \neq 0$, for all $w_i \in W$. $\Delta_\mu^{d,W}$ satisfies **(IC2)** if and only if $w_1 \neq 0$, where $w_1 \in W$.*

(IC6b) is equivalent to Strong Pareto, which can be translated as: if $\forall i \ d(\omega', K_i) \leq d(\omega, K_i)$ and $\exists j \ d(\omega', K_j) < d(\omega, K_j)$, then $\omega' < \omega$. Thus, The existence of a $w_j = 0$ is sufficient to falsify this condition. The reason why **(IC2)** is not always true comes from the fact that even if an outcome does not have a consensus between agents, it can still be a choice of the merging operator. Consequently, any OWA merging operator $\Delta_\mu^{d,W}$ is a pre-IC merging operator when it satisfies the conditions of Theorem 2.

Corollary 1 *If $w_i = \frac{1}{n}$, for all $w_i \in W$ and $W = |n|$, then $\Delta_\mu^{d,W}$ satisfies **(Maj)**, **(IC5)** and **(IC6)**. If $w_1 = 1$ and $w_i = 0$, for all $i \neq 1$, then $\Delta_\mu^{d,W}$ satisfies **(IC5)**. If $w_n = 1$ and $w_i = 0$, for all $i \neq n$, then $\Delta_\mu^{d,W}$ satisfies **(IC5)**.*

It is not known if have two-sided conditions for **(IC5)**, **(IC6)** and **(Maj)** in relation with OWA merging operators. The same holds for **(Arb)**.

Theorem 3 *Let d be a distance measure, ω an interpretation and $m = max(\{d(\omega, \omega') \mid \omega, \omega' \in \Omega\})$. If $w_1 > (m-1)w_2$, then $\Delta_\mu^{d,[w_1,w_2]}$ satisfies **(Arb)** .*

When we refer to **(PD)**, we have the following result:

Theorem 4 *$\Delta_\mu^{d,W}$ satisfies **(PD)** if and only if $w_1 > w_2 > w_3 > \cdots > w_n$, for $W = [w_1, \ldots, w_n]$.*

In other words, if we are giving more priority to the worst case, and the weights are successively decreasing for the next cases, we are guaranteeing a more balanced merging for the group.

OWA operators are powerful enough to simulate the $leximax$ ordering. In (Yager 1997) it was defined an OWA which simulates it, and following this operator we can define a $leximax$ like merging operator.

Definition 3 (**$leximax$ like OWA Operators**) *We say f_{W_δ} is a $leximax$ like OWA Operator if their weights are defined as $W_\delta = [w_1, \ldots, w_n]$, such that $\delta \in]0,1]$, $w_i = \frac{\delta^{i-1}}{(1+\delta)^i}$, for $i \neq n$; and $w_n = \frac{\delta^{n-1}}{(1+\delta)^{n-1}}$.*

The idea of this operator is to give the highest weight to the highest value of a vector and this weight decreases to the consequent values. Depending of the value of δ, the difference of weights are so large that the operator gives an absolute priority to the highest value than the other values of the vector.

Theorem 5 *Let d be a distance measure, ω an interpretation and $m = max(\{d(\omega, \omega') \mid \omega, \omega' \in \Omega\})$. Consider $W = [w_1, \ldots, w_n]$, where $w_i = \frac{\delta^{i-1}}{(1+\delta)^i}$, for $i \neq n$ and $w_n = \frac{\delta^{n-1}}{(1+\delta)^{n-1}}$. If $\delta \leq \frac{1}{m}$, then Δ_μ^{d,W_δ} satisfies **(HE)**.*

In other words, Δ_μ^{d,W_δ} is not equivalent to $\Delta_\mu^{d,leximax}(E)$, but as every belief set E has a finite number of belief bases, which are also finite, it is possible to find a δ' such that $\Delta_\mu^{d,W_{\delta'}}(E) \equiv \Delta_\mu^{d,leximax}(E)$.

Conclusion

The choice of the weights plays a fundamental role in the relation of the satisfaction of some IC logical postulates. In general, logical postulates as **(IC2)**, **(IC5)**, **(IC6)**, **(Maj)** and **(Arb)** are not satisfied by OWA merging operators. We showed that when some conditions are met, these properties can be satisfied. Furthermore, we still explored two egalitarian conditions: Pigou-Dalton and Hammond Equity. We also proved these conditions can be satisfied when some restrictions are applied to the weights. Therefore, OWA merging operators are powerful enough to represent IC and pre-IC merging operators.

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