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Super Altruistic Hedonic Games

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Abstract

Hedonic games are coalition formation games in which agents' utility depends only on their own coalition. The introduction of Altruistic Hedonic Games increased the expressive potential of Hedonic Games by considering the utility of each of the agent's friends within the coalition. We introduce *Super Altruistic Hedonic Games (SAHGs),* in which an agent's utility may depend on the utility of all other agents in the coalition, weighted according to their distance in the friendship graph. We establish the framework for this new model and investigate the complexity of multiple notions of stability. We show that SAHGs generalize Friend-oriented Hedonic Games, Enemy-oriented Hedonic Games, and selfish-first Altruistic Hedonic Games, inheriting the hardness results of these games as minimum upper complexity bounds. We also give SAHGs that have neither Nash stable nor strictly core stable partitions.

Introduction

Consider the process of choosing where to live. Much has been written (in the RecSys literature, preferences, etc.) about how to choose the right house or apartment, even the right roommates for a stable configuration. Let us consider the choice of *neighbors*, perhaps in a setting where students are choosing their dormitories/hostels. We can see the partitioning of students into living units (floors, buildings, etc.) as an hedonic game. It is clear that we value our friends' happiness with the living situation, as we will hear about it from them; our enemies' happiness could be assumed to also affect how they treat us. (If we stopped there, we would be modeling evaluation as a *Altruistic Hedonic Game*.) More generally we can also argue that our friends' friends' happiness will affect our friends', and thus indirectly, our own, and that this continues out friendship chains, with decreasing (or at least, non-increasing) effect as we increase the social distance from ourselves.

If we were building intranets, a node could evaluate the quality of the local network in terms of the bandwidth to reachable nodes. However, it would also need to take into account the quality of more distant connections, if it hopes to have its packets relayed. There are many other applications in which agents care not only about immediate connections, but also those farther away. We introduce a family of hedonic games that model such broad evaluations of coalitions: the Super Altruistic Hedonic Games.

Related Work

SAHGs are a natural extension of Altruistic Hedonic Games (AHGs) wherein agents consider the preferences of other agents (Nguyen et al. 2016). In AHGs, agents only consider the preferences of their friends. In SAHGS, agents consider the preferences of all agents in their coalition. In AHGs, friends are assigned fixed weights. In SAHGs, the weights assigned to friends and enemies are not fixed, and the preferences of all agents in a coalition are considered, often taking advantage of indirect relationships such as friends of friends to adjust weights. (Note that friendship is not transitive: a friend of a friend could be our enemy.)

Social Distance Games (SDGs) are a class of coalition formation games wherein an agent's utility is a measure of their closeness, or social distance, from the other members of their coalition (Brânzei and Larson 2011). SDGs have certain similarities to SAHGs, but we believe that SAHGs can better model realistic human interactions by combining the notion of social distance with the consideration of others' preferences proposed in AHGs.

As we demonstrate later, SAHGs generalize Friends and Enemies-oriented Hedonic Games (FHGs and EHGs) (Dimitrov et al. 2006). In the former, agents seek to find coalitions that maximize the number of friends with a secondary goal of minimizing the number of enemies. In the latter, minimizing the number of enemies is the primary goal, while maximizing the number of friends becomes secondary. Recent work has investigated the impact that neutral agents have on these games, defining a neutral agent as one that is neither friend nor enemy (Ohta et al. 2017). It was shown that permitting neutral agents in EHGs allows for games that have no core stable partition (Ohta et al. 2017). Core stable partitions are still guaranteed to exist in FHGs with neutral agents; however, strict-core stable partitions are not (Ohta et al. 2017). The proofs of these findings cannot be readily translated to SAHGs, because SAHGs do not allow neutral agents. Neutral agents could be modeled as graph-based games by labeling appropriate edges as neutral, but SAHGs are focused on simple graph-based models, so the addition of neutral edges is beyond the scope of this paper.

There are graph-related hedonic games that depend on edge-weighted graphs. For instance, $\mathcal{B}$ and $\mathcal{W}$ games are a category of hedonic games in which an agent's utility is defined by the agents in their coalition that they rate as the best or the worst, respectively (Cechlárová and Romero-Medina 2001). While these games fall into the category of hedonic games, we don't believe SAHGs can generalize $\mathcal{B}$ or $\mathcal{W}$ games. Similarly, we do not believe that either $\mathcal{B}$ or $\mathcal{W}$ games can generalize SAHGs. This is due to the differences between $\mathcal{B}$ and $\mathcal{W}$ games and SAHGS, such as the former two categories assuming each agent can assign a unique value to each other agent, while SAHGs restrict agents to placing others into one of two categories. Additionally, $\mathcal{B}$ and $\mathcal{W}$ games do not consider the preferences of others as SAHGs do.

Preliminaries

Below, we outline three types of cooperative games with non-transferable utility, specifically coalition formation games. In each type, a game G consists of

1. N, a finite set of n agents, with
2. preference set $P = \{P_i : i \in N\}$, where P_i is the preference of each agent i over partitions of N into coalitions.

Depending on the type of game, P may exhaustively list each individual's preferences or provide a succinct representation from which preferences are derived.

Definition 1. *(Banerjee, Konishi, and Sönmez 2001; Bogomolnaia and Jackson 2002)* ***Hedonic games*** *are coalition formation games with nontransferable utility wherein players are concerned only with their own coalition. This inherently self-interested means of determining utility makes such games hedonic in nature.*

Let $\mathcal{N}_i$ be the set of possible coalitions containing agent $i \in N$. A preference ordering of $\mathcal{N}_i$ is derived from the preference set $P_i \in P$. A solution for a game is a partition π, which is contained in the set of all distinct partitions Γ. Each player $i \in N$ ranks each partition $\pi \in \Gamma$ based on the coalition to which they belong.

Hedonic games are a broad category, so it can be useful to define sub-categories that exhibit certain interesting or useful properties. Friend and enemy-oriented hedonic games are two categories.

Definition 2. *A* ***Friend-oriented Hedonic Game (FHG)*** *(Dimitrov et al. 2006) is characterized by agents assigned values to each other based on whether they view each other as a friend or an enemy. FHGs are often represented by graphs where an edge from some agent $i \in N$ to another agent $j \in N$ indicates that i regards j as a friend. Lack of an edge from i to j indicates that i regards j as an enemy. Utility for each agent is the sum of values they assign to other agents, friends being assigned a value of n while enemies are valued at -1.*

Enemy-oriented Hedonic Games (EHGs) *are based on the same principles as FHGs, but friends are instead assigned a value of 1, while enemies are assigned a value of $-n$.*

Altruistic hedonic games are another sub-category that expands on the ideas in FHGs and EHGs and is a major inspiration for the work done in this paper.

Definition 3. *(Nguyen et al. 2016) An* ***altruistic hedonic game (AHG)*** *is a hedonic game in which agents derive utility from both their own basic preferences and those of any friends in the same coalition.*

Let each agent $i \in N$ have utility u_i, and let i partition other agents into friends and enemies, given by F_i, E_i. Three levels of altruism are considered in AHGs: ***selfish-first***, ***equal treatment***, *and* ***altruistic first***. *The function used to determine an agent's utility depends on their altruism level and on pre-utility preference values calculated as the utility agents would have in a friends-oriented hedonic game based on the same graph ($n|C \cap F_i| - |C \cap E_i|$). Two of these functions utilize a weight parameter of $M = n^5$ to ensure that one of the terms in the equation dominates the other. This weight value is the smallest whole number exponent of n which guarantees this for both equations that make use of M. Definitions for each altruism level and their utility functions are outlined below:*

1. ***Selfish-First:*** *agents prioritize their own preferences, but use the preferences of others to break ties.*

$$u_i = M(n|C \cap F_i| - |C \cap E_i|) + \sum_{a \in C \cap F_i} \frac{n|C \cap F_a| - |C \cap E_a|}{|C \cap F_i|}$$

2. ***Equal Treatment:*** *all preferences are treated equally.*

$$u_i = \sum_{a \in C \cap (F_i \cup \{i\})} \frac{n|C \cap F_a| - |C \cap E_a|}{|C \cap (F_i \cup \{i\})|}$$

3. ***Altruistic First:*** *agents prioritize the preferences of others, but use their own preferences to break ties.*

$$u_i = n|C \cap F_i| - |C \cap E_i| + M \cdot \sum_{a \in C \cap F_i} \frac{n|C \cap F_a| - |C \cap E_a|}{|C \cap F_i|}$$

We assume familiarity with the complexity classes P and NP, but also reference two lesser-known complexity classes, DP and Θ_2^p.

Definition 4. *(Papadimitriou and Yannakakis 1982) The complexity class* ***DP*** *contains languages defined as the difference between two languages in NP.*

For example, let C be an NP-complete language, and let $L = \{\langle c_1, c_2 \rangle : c_1 \in C \wedge c_2 \notin C\}$. Then $L = \{C \times \Sigma^*\} \setminus \{\Sigma^* \times C\}$ (where Σ^* is the set of all strings over the alphabet used to define C).

Definition 5. *Complexity class Θ_2^p is an alternative name for $P^{NP[log]}$ (Hemachandra 1987). Games in this class are solvable by a P machine that can make $O(\log n)$ queries to an NP oracle.*

Stability and Optimality

One of the major topics of hedonic games is *stability*, the idea that a partition will not be disrupted by individuals rejecting their assigned coalitions and moving to other coalitions. There are many sets of constraints placed on such disruptions, such as the number of agents that can move simultaneously; whether all moving agents must see an increase in utility; whether agents left behind by movers must see their utility increase, or whether agents being joined by movers must see their utility improve.

Optimality, the notion of finding a utility-maximizing partition, is another major topic of hedonic games. Notions of optimality are subject to constraints which clarify what is being optimized, such as whether individual or collective (egalitarian or utilitarian) utility is being optimized, or whether there is a solution that improves or holds fixed everyone's utility — if not, the solution is *Pareto efficient*.

We next define notions related to stability that are referenced in the rest of the paper. In these definitions π is a partition composed of a set of k disjoint coalitions $\{C_1, C_2, ...C_k\}$.

- *Nash Stability* (Bogomolnaia and Jackson 2002): $\forall i \in N$ and $\forall C \in \pi : C \neq \pi(i)$ we have $\pi(i) \succeq_i C \cup \{i\}$.
- *Individual Stability* (Bogomolnaia and Jackson 2002): $\forall i \in N$ and $\forall C \in \pi \cup \{\emptyset\} : C \neq \pi(i)$: $\pi(i) \succeq_i C \cup \{i\}$ or $\exists j : C \succ_j C \cup \{i\}$. Permission must be received from all existing coalition members before a new agent can join.
- *Contractual Individual Stability* (Bogomolnaia and Jackson 2002): $\forall i \in N$ and $\forall C \in \pi \cup \{\emptyset\} : C \neq \pi(i)$:
  $\pi(i) \succeq_i C \cup \{i\}$, $\exists j : C \succ_j C \cup \{i\}$, or
  $\exists k : \pi(i) \succ_k \pi(i) \setminus \{i\}$.
- *Wonderful Stability* (Woeginger 2013): $\forall C \in \pi : C$ is a maximal (non-extendable) clique.
- *Strictly Popular* (Nguyen et al. 2016): partition π beats all other $\pi' \neq \pi$ in pairwise comparisons
$$|\{i \in N | \pi(i) \succ_i \pi'(i)\}| > |\{i \in N | \pi'(i) \succ_i \pi(i)\}|$$
- *Blocking coalition* (Rothe 2016): A coalition C blocks partition π if $\forall i \in C : C \succ_i \pi(i)$.
- *Weakly blocking coalition* (Rothe 2016): A coalition C weakly blocks partition π if $\forall i \in C : C \succeq_i \pi(i)$ and $\exists j : C \succ_j \pi(j)$.
- *(Strict) Core Stability* (Rothe 2016): no (weakly) blocking coalition exists.

Super Altruistic Hedonic Games

AHGs introduce some interesting ideas by incorporating the preferences of others into utility computations in a polynomially computable fashion. The three levels of altruism provide a means to vary the degree to which agents consider the preferences of others, while also providing bounds on the weights needed to ensure the dominance of one term in the utility equation. However, only considering the preferences of friends and three variations of altruism limits the preferences and degrees of altruism that can be represented. We introduce Super Altruistic Hedonic Games in order to broaden the scope of representation.

Super Altruistic Hedonic Games(SAHGs) are a family of extensions to AHGs, parameterized by the ratio of selfishness and others' preferences, and by the relative weight of the preferences of friends, friends of friends, etc.

Definition 6. *A SAHG instance is defined by a set of agents N and a preference set P. Let (a, g, M, L) be non-negative weights. The values a and g are the weights associated with friends and enemies, respectively. The values M and L are the weights associated with personal preference and others' preferences. The function $D(i, j)$, representing the weight i gives to j's preferences, is a polynomial-time computable, non-increasing function of the graph distance $d_G(i, j)$ in the friendship graph G.*

Let the number of other agents in coalition C_i be $h_i = |C_i \setminus \{i\}|$. For each agent $i \in N$, let that agent's base preference be $b_i = a|C_i \cap F_i| - g|C_i \cap E_i|$, and let their utility be
$$u_i = Mb_i + L \sum_{j \in C_i \setminus \{i\}} \frac{D(i, j) \cdot b_j}{h_i}.$$
*(If $C_i = \{i\}$ then the sum is set to 0.) The default definition of D is the inverse graph distance function: for any pair of agents $i, j \in N : i \neq j$, let d_{ij} be the shortest path distance between them, then let $D(i, j) = 1/d_{ij}$. The **total utility** of a partition π is given by $U_T = \sum_{i \in N} u_i$.*

Proposition 1 follows from the definition of SAHGs.

Proposition 1. *SAHGs generalize several graph-based hedonic games.*

- *A Friends-oriented Hedonic Game (FHG) is a SAHG with parameters $(a, g, M, L) = (n, 1, 1, 0)$, and an Enemies-oriented Hedonic Game (EHG) is a SAHG with parameters $(1, n, 1, 0)$. (Because $L = 0$, it does not matter how we define D.) Thus, SAHGs inherit all hardness results for FHGs and EHGs as lower bounds on hardness.*
- *SAHGs also model Altruistic Hedonic Games under the selfish-first criterion ($(a, g, M, L) = (n, 1, n^5, 1)$ and $D(i, j) = 1$ if $j \in F_i$ and $D(i, j) = 0$ if $j \notin F_i$).*
- *If $D \equiv 1$ and $(a, g, M, L) = (n, 1, 1, 1)$ then we capture the notion of a friend-oriented hedonic game on the transitive closure of the friendship graph.*

Proposition 2. *If a coalition comprises a single clique, C, then individual utilities are given by a linear function of the number of agents and coalition utility is defined by a polynomial function of the number of agents.*

Proof. We first recall that the base preference of each agent $i \in N$ is given by $b_i = a|C_i \cap F_i| - g|C_i \cap E_i|$ where C_i is the coalition to which i belongs, and that $h_i = |C_i \cap F_i| + |C_i \cap E_i|$ defines the number of agents in $C_i \setminus \{i\}$. Next recall that each agent $i \in N$ has utility given by
$$u_i = Mb_i + L \sum_{j \in C_i \setminus \{i\}} \frac{D(i, j) \cdot b_j}{h_i}.$$
The total utility of a partition is defined by $U_T = \sum_{i \in N} u_i$.

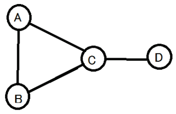

Figure 1: Unequal Cliques

Now we define the total utility of a coalition as $U_C = \sum_{i \in C} u_i$. Because C is a clique, we know that $\forall i, j \in C, i \neq j$ $D(i,j) = 1$. We also know that all $i \in C$ have $h_i = |C| - 1$ and $b_i = a(|C| - 1)$. We use this to calculate

$$u_i = M \cdot a(|C| - 1) + L \sum_{j \in C_i \setminus \{i\}} \frac{a(|C| - 1)}{1(|C| - 1)},$$

which simplifies to $u_i = (M + L) \cdot a(|C| - 1)$.

The total utility of the coalition is $U_C = \sum_{i \in C} u_i$, which simplifies to $U_C = (M + L)(a(|C|^2 - |C|))$. Thus, we have demonstrated that, given a coalition C comprised of a single clique, the individual utility is a linear function of $|C|$ and the coalition utility is a polynomial function of $|C|$. $\square$

Proposition 3. *Different partitions of a set of agents into cliques may have different utilities.*

Proof. Consider a game based on Figure with parameters $(a, g, M, L) = (1, 1, 1, 1)$. For each $i \in N$ we have:

- $b_i = |C_i \cap F_i| - |C_i \cap E_i|$
- $u_i = b_i + \sum_{j \in C_i \setminus \{i\}} \frac{D(i,j) \cdot b_j}{h_i}$.

Consider two partitions:
$\pi_1 = \{\{A, B, C\}, \{D\}\}$ and $\pi_2 = \{\{A, B\}, \{C, D\}\}$.
In π_1, we have $b_A = b_B = b_C = 2$ and $b_D = 0$. We also have $u_A = u_B = u_C = 4$ and $u_D = 0$ and $U_T(\pi_1) = 12$. In π_2, we have $b_A = b_B = b_C = b_D = 1$ and $u_A = u_B = u_C = u_D = 2$. Thus, $U_T(\pi_2) = 8$.

Since we have two partitions into cliques with different total utility values, we can conclude that partitioning agents into cliques does not ensure a consistent total utility. However, given partitions π_3 and π_4 dividing agents into equal numbers of cliques of each size, the total utilities of the two partitions will be the same. $\square$

Proposition 4. *For all parameter values, for all stability notions considered in this paper, there exist SAHGs with stable partitions.*

Proof. Let G be the SAHG with structure given by a graph with n nodes and no edges with parameters (a, g, M, L). For the partition of singletons, each agent i has utility $u_i = 0$. Since there are no edges in the graph, no agent would benefit from forming a coalition with any other agent or set of agents, so the partition of singletons is stable. $\square$

Theorem 1. *Not all SAHGs have strictly core stable partitions, even when friend and enemy relationships are symmetric.*

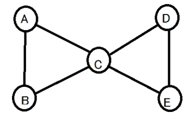

Figure 2: Game with no strictly core partition

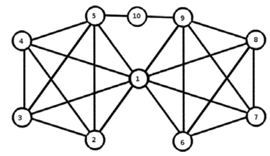

Figure 3: Game with no *Nash stable* partition

Proof. Consider a game based on Figure with parameters $(a, g, M, L) = (1, 1, 1, 1)$. For each agent $i \in N$, we have:

- $b_i = |C_i \cap F_i| - |C_i \cap E_i|$
- $u_i = b_i + \sum_{j \in C_i \setminus i} \frac{D(i,j) \cdot b_j}{h_i}$.

This game contains two equal-sized cliques connected by a single intermediate agent, C. The grand coalition is weakly blocked by each of $\{A, B, C\}$ and $\{C, D, E\}$. If one of these weakly blocking coalitions splits off from the grand coalition, we either have
$\pi_1 = \{A, B, C\}, \{D, E\}$ or $\pi_2 = \{A, B\}, \{C, D, E\}$.
π_1 is weakly blocked by $\{C, D, E\}$ and π_2 is weakly blocked by $\{A, B, C\}$.

The utility of A and B is maximized in $\{A, B, C\}$, while $\{C, D, E\}$ maximizes the utility of C and D. The utility of agent C is maximized by the grand coalition and by $\{A, B, C\}$ and $\{C, D, E\}$. As such, all possible partitions are weakly blocked by $\{A, B, C\}$, $\{C, D, E\}$, or both. Thus there is no strictly core stable partition. $\square$

Theorem 2. *Not all SAHGs have Nash stable partitions, even when friend and enemy relationships are symmetric.*

Proof. Let G be the SAHG with structure given in Figure , and weight parameters $(a, g, M, L) = (1, 1, 1, 3)$. This gives us:

- $b_i = |C_i \cap F_i| - |C_i \cap E_i|$
- $u_i = b_i + 3 \sum_{j \in C_i \setminus i} \frac{D_(i,j) \cdot b_j}{h_i}$.

This game has two equal-sized cliques which are connected to each other through two intermediate agents. The first connecting agent, agent 1, is connected to all agents in both cliques. The second connecting agent, agent 10, is connected to a single agent in each clique and is not connected to agent 1. The first clique is composed of agents 1–5 and the second of agents 1, 6–9.

Because the only member common to both cliques is agent 1, it is reasonable to expect that no stable coalition containing one clique will contain any members from the other, except for agent 1. If members from two cliques form into coalitions which do not include agents 1 and 10, then these two remaining agents would prefer to remain as singletons rather than forming a two-person coalition with each other. In this case, the utility of an agent in one of the two clique coalitions is 12, while the utility of agents 1 and 10 are zero since they are singletons. This describes partition $\pi_1 = \{\{1\}, \{2, 3, 4, 5\}, \{6, 7, 8, 9\}, \{10\}\}$ with total utility $U_T = 96$.

The partition π_1 is unstable, because agent 1 can improve their utility by joining one of the two clique coalitions. Since agent 1 is connected to all agents in both cliques, its joining either coalition will increase the size of the clique by 1, increasing the utility of all agents in the coalition from 12 to 16. Agent 1 is indifferent between the two cliques. This presents two possible partitions
$\pi_2 = \{\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9\}, \{10\}\}$ and
$\pi_3 = \{\{2, 3, 4, 5\}, \{1, 6, 7, 8, 9\}, \{10\}\}$,
each of which has total utility $U_T = 128$.

Both π_2 and π_3 are also unstable because agent 10 can also improve its own utility by joining a coalition. If agent 10 chooses to join the coalition that agent 1 did not, it derives utility $u_{10} = 3.25$, while it derives utility $u_{10} = 3.6$ if it joins the same coalition as agent 1. Thus, agent 10 prefers to join whichever coalition agent 1 joined, which results in either $\pi_4 = \{\{1, 2, 3, 4, 5, 10\}, \{6, 7, 8, 9\}\}$ or $\pi_5 = \{\{2, 3, 4, 5\}, \{1, 6, 7, 8, 9, 10\}\}$. The total utility of this new partition is $U_T = 104$.

Still, π_4 and π_5 are unstable because agent 1 can improve its utility by leaving the current coalition to join the other clique, thereby restoring its utility to 16. This gives either $\pi_6 = \{\{2, 3, 4, 5, 10\}, \{1, 6, 7, 8, 9\}\}$ or $\pi_7 = \{\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10\}\}$. In these partitions, the total utility is $U_T = 110.5$. However, π_6 and π_7 are unstable since agent 10 can improve its utility by following agent 1, which creates a cycle of four partitions, none of which are Nash stable. Thus we conclude that there is no Nash stable partition for this game, and, by extension, that not all SAHGs are guaranteed to have Nash stable partitions. □

Theorem 2 contrasts with existing results for AHGs, which always have Nash stable partitions (Nguyen et al. 2016), and FHGs, which always have strictly core stable partitions (Dimitrov et al. 2006).

Notice that the game in the proof of Theorem 2 has core stable partitions: $\{\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9\}, \{10\}\}$ and $\{\{2, 3, 4, 5\}, \{1, 6, 7, 8, 9\}, \{10\}\}$. The 5-member cliques weakly block the opposing partition, but there are no coalitions that block either partition. Additionally, agent 10 would not be accepted in either coalition, since its presence decreases the utility of every other member in the coalition.

Proposition 5 follows from previous work by Ballester (2004) proving that all hedonic games have contractually individually stable partitions.

Proposition 5. *Contractually individually stable partitions are guaranteed to exist for SAHGs.*

Computational Complexity

Proposition 6. *Computing the utility of a partition for a SAHG is in P.*

Proof. Consider a partition π of some game G. The steps to evaluate the partition are:

1. $\forall i \in N$ and $\forall j \in \pi(i)$ compute $D(i, j)$
2. $\forall i \in N$ compute h_i and b_i
3. $\forall i \in N$ compute u_i
4. compute $U_T(\pi)$.

We assume that intermediate values are computed once and stored.

In the default case where $D(i, j)$ is the graph distance between i and j, we can use the Floyd-Warshall algorithm to compute this distance for all $(i, j) \in N \times N$ in time $\mathcal{O}(n^3)$ (Cormen et al. 2009), otherwise, it is $\mathcal{O}(n^2)t(n)$, where $t(n)$ is the time needed to compute any $D(i, j)$ for a SAHG of size n. We compute h_i and b_i in time $\mathcal{O}(n^2)$ by checking each entry in $\pi(i)$ against the lists F_i and E_i. Computing h_i and b_i for all $i \in N$ requires time $\mathcal{O}(n^3)$. Calculating u_i requires time $\Theta(|\pi(i)|) < \mathcal{O}(n)$. So the time required to compute u_i for all $i \in N$ is $\mathcal{O}(n^2)$. $U_T(\pi)$ can then be computed in time $\mathcal{O}(n)$. The overall time required to evaluate a partition is $\mathcal{O}(n^3)$. Thus a partition of a Super Altruistic Hedonic Game can be evaluated in polynomial time. □

Proposition 7. *Deciding whether a partition is Nash stable is in time $\mathcal{O}(n^2 e(n))$, where $e(n)$ is the time needed to evaluate the utility of a coalition.*

Proof. Consider a partition π of some game G. To determine if π is Nash stable, $\forall i \in N$ and $\forall C \in \pi : C \neq \pi(i)$ we compare $u_i(\pi(i))$ with $u_i(C \cup \{i\})$. If there exists no (i, C) such that $u_i(C \cup \{i\}) > u_i(\pi(i))$, then π is Nash stable.

There are at most n coalitions in π in the case of the partition of singletons, and for each $C \in \pi$, n utility values must be computed. At most n^2 utility values must be computed to determine if π is Nash stable. Determining if a partition π is Nash stable requires time $\mathcal{O}(n^2 e(n))$ where $e(n)$ is the time needed to compute the utility of a coalition. □

We have previously demonstrated that FHGs, EHGs, and selfish-first AHGs are generalized by SAHGs. As a result, SAHGs inherit the complexity results of these games as lower bounds. These results are outlined in Corollary 1.

Corollary 1.

- *Determining if strictly popular partitions exist in SAHGS is coNP-hard (Nguyen et al. 2016).*
- *Verifying a partition is strictly popular is coNP-hard (Nguyen et al. 2016).*
- *Determining if strictly core stable partitions exist is DP-hard (Rey 2016; Rey et al. 2016).*
- *Verifying a partition is (strictly) core stable is coNP-hard (Woeginger 2013).*
- *Determining if wonderfully stable partitions exist is DP-hard (Rey 2016; Rey et al. 2016).*

Conclusions and Open Questions

We introduce SAHGs as an extension of the ideas behind AHGs, introduced by Nguyen et al. (2016). We show that SAHGs generalize several graph-based hedonic games: FHGs, EHGs, and AHGs under the selfish-first criterion in proposition 1. What distinguishes SAHGs from the games they generalize is the consideration of the preferences of all other agents in one's coalition. This difference allows SAHGs to better model partitioning problems with a larger scope than roommate assignment problems.

We examine several properties of SAHGs in propositions 2–4. We prove that stable partitions may not exist, even when friendship relations are symmetric in theorems 1 and 2. We show that the total utility of a partition can be computed in polynomial time in proposition 6. Proposition 7 demonstrates how Nash stability can be verified in polynomial time. Corollary 1 clarifies lower bounds on complexity that SAHGs inherit from games they generalize.

A Probably Approximately Correct (PAC) learning model is intended to find good function approximations. This model has previously been applied to several varieties of hedonic games (Sliwinski and Zick 2017), for instance, to PAC learn stability. We conjecture that SAHGs are also PAC learnable.

Future work will address the complexity of optimal partition algorithms for SAHGs, and algorithms for *finding* stable partitions when they exist.

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