Proceedings of the Fourteenth International Conference on Principles of Knowledge Representation and Reasoning

Belief Change and Semiorders

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Abstract

A central result in the AGM framework for belief revision is the construction of revision functions in terms of *total preorders* on possible worlds. These preorders encode comparative plausibility: $r \prec r'$ states that the world r is at least as *plausible* as r'. *Indifference* in the plausibility of two worlds, r, r', denoted $r \sim r'$, is defined as $r \not\prec r'$ and $r' \not\prec r$. Herein we take a closer look at plausibility indifference. We contend that the transitivity of indifference assumed in the AGM framework is not always a desirable property for comparative plausibility. Our argument originates from similar concerns in *preference modelling*, where a structure weaker than a total preorder, called a *semiorder*, is widely consider to be a more adequate model of preference. In this paper we essentially re-construct revision functions using semiorders instead of total preorders. We formulate postulates to characterise this new, wider, class of revision functions, and prove that the postulates are sound and complete with respect to the semiorder-based construction. The corresponding class of contraction functions (via the Levi and Harper Identities) is also characterised axiomatically.

Introduction

Plausibility rankings lie at the heart of modelling *belief change*. They take different forms depending on the approach employed and the type of belief change encoded – see (Peppas and Williams 1995) – ranging from *preorders* on possible worlds (Katsuno and Mendelzon 1991), to *epistemic entrenchments* (Gardenfors and Makinson 1988), to *ordering on remainders* (Alchourron, Gardenfors, and Makinson 1985). The essence however in all such forms is the same: plausibility ranking are used to determine the most plausible objects (worlds, sentences, or remainders respectively), among the alternatives in view of the epistemic input and the given type of belief change. That is, whenever during belief change, one is forced to choose between α or β, the most plausible of the two will be selected (as determined by the corresponding plausibility ranking).

An underlying assumption employed in the prevalent *AGM framework*[1] for belief change, is that *indifference of comparative plausibility is transitive*. To better illustrate our point, let us focus on plausibility rankings $\preccurlyeq$ on possible worlds, called *faithful preorders* in (Katsuno and Mendelzon 1991).

Suppose that two distinct possible worlds w_1, w_2 are equally plausible (or implausible) relative to the agents' current belief state K. We shall denote this by $w_1 \sim w_2$; formally, $w_1 \sim w_2$ iff $w_1 \not\prec w_2$ and $w_2 \not\prec w_1$, where $\prec$ denotes the strict part of $\preccurlyeq$. Suppose now that w_2 is equally plausible to a third world w_3; i.e. $w_2 \sim w_3$. In the AGM framework we automatically conclude that $w_1 \sim w_3$.

Economists on the other hand are more cautious. It has long been acknowledged in the area of *preference modelling* that transitivity is not always a natural property for indifference of preference. The following quote from (Luce 1956) illustrates the problem:

> *"Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes (this should not be difficult). Now prepare 401 cups of coffee with* $(1 + i/100) \cdot x$ *grams of sugar,* $i = 0, 1, \cdots, 400$, *where x is the weight of one cube of sugar. It is evident that he will be indifferent between cup i and cup* $i + 1$, *for any i, but by choice he is not indifferent between* $i = 0$ *and* $i = 400$."

The above example, along with further arguments made in (Luce 1956), support the view that rational agents tend to discriminate between two alternatives α, β only when their difference[2] exceeds a certain *threshold*. As noted by Armstrong in (Armstrong 1950),

> *"The nontransitiveness of indifference must be recognized and explained on [sic] any theory of choice, and the only explanation that seems to work is based on the imperfect powers of discrimination of the human mind whereby inequalities become recognizable only when of sufficient magnitude."*

One can imagine similar scenarios in the context of belief change. To use (an adequately adjusted variation of) a

[1]The AGM framework is the formal framework introduced by *Alchourron, Gardenfors, and Makinson* in (Alchourron, Gardenfors, and Makinson 1985), and further developed by many others (see (Peppas 2008)) for the study of belief change. We shall review its main constituents later in the paper.

[2]Or more accurately, the difference of their respective utilities.

well known example, called *the bald man paradox* – see (Ludwig and Ray 2002) – suppose that our agent, Myrto, believes that her grandfather Speros, whom she never met, had a full head of hair. It is therefore reasonable to assume that the possible world w_{5000} in which her grandfather has 5000 hairs is more plausible to Myrto than the world w_{50} in which Speros has only 50 hairs. On the other hand, it is also reasonable to assume that Myrto is indifferent between worlds, like w_{4657} and w_{4656}, which differ only in that Speros has a single hair less in the latter. If indifference was transitive (as it is the case in the AGM framework), with 4050 applications of transitivity we would derive that Myrto is indifferent between w_{5000} and w_{50}, which of course it is not true.

Indifference between alternatives with fine variations may also be conscious, and not due to 'imperfect powers of discrimination". Ackerman illustrates this point with the following example, (Ackerman 1994):

> *"It is entirely plausible to suppose that an instructor would be indifferent to having the number of students in his seminar be 6 vs. 7, 7 vs. 8, etc., without being indifferent to having it be 6 vs. 15; he might consider 15 students too many for a seminar. But he can certainly discriminate between 6 and 7 students or 7 and 8, etc."*

Another reason for the intransitivity of preference indifference (and likewise, for the intransitivity of plausibility indifference) is the *multi-dimensionality of alternatives*. To take a two-dimensional example from (Fishburn 1970a):[3]

> *"You are going to buy a car. You have no definite preference between (Ford, at \$2600) and (Chevrolet, at \$2700), and also have no definite preference between (Ford, at \$2600) and (Chevrolet, at \$2705). However, (Chevrolet, at \$2705) < (Chevrolet, at \$2700)."*

Considerations like these led Luce to develop in (Luce 1956) a new structure, called a *semiorder*, which has since become widely accepted in the community of mathematical psychology as an adequate model for human preference (see for example, (Fishburn 1970b), (Jamison and Lau 1973), (Pirlot and Vincke 1997), (Rabinovitch 1977), (Rubinstein 2012)).

Herein, and for similar reasons, semiorders are imported in Belief Change; i.e. we re-construct the basic models and results of the AGM framework using semiorders (instead of total preorder) to model comparative plausibility. In particular, we formulate postulates that characterize the (wider) class of revision functions induced from semiorders, and we prove their correctness (i.e. we show that the postulates are *sound* and *complete* with respect to the semiorders-based construction). Moreover, we characterize axiomatically the class of *contraction functions* corresponding, via the Levi and Harper Identities, to semiorder-based revision functions.

The paper is structured as follows. In the next three sections we introduce some notation, we briefly review the AGM framework, and we recall the main definitions and results on semiorders. Following that there are two sections containing the main results of the paper. Finally, in the last two sections we discuss related works and make some concluding remarks.

[3] Notice that in this example "better" items appear higher in the $<$-ordering. This is essentially the reverse of the case with faithful preorders where the more plausible a world is, the lower it appears in the preorder.

Formal Preliminaries

Throughout this paper we work with a finite set of propositional variables P. We define L to be the propositional language generated from P (using the standard boolean connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$ and the special symbols $\top, \bot$) and governed by classical propositional logic.

For a set of sentences Γ of L, we denote by $Cn(\Gamma)$ the set of all logical consequences of Γ, i.e., $Cn(\Gamma) = \{x \in L: \Gamma \models x\}$. We shall often write $Cn(x_1, \ldots, x_n)$, for sentences $x_1, \ldots, x_n$, as an abbreviation of $Cn(\{x_1, \ldots, x_n\})$. For any two sentences x, y we shall write $x \equiv y$ iff $Cn(x) = Cn(y)$.

A theory K of L is any set of sentences of L closed under $\models$, i.e., $K = Cn(K)$. We shall denote the set of all theories of L by $\mathcal{T}$. We define a *possible world* r (or simply a *world*), to be a consistent set of literals such that for any propositional variables $x \in P$, either $x \in r$ or $\neg x \in r$. We will often identify a world r with the conjunction of its literals, leaving it to the context to resolve any ambiguity (for example in "$\neg r$", r is a sentence, whereas in "$r \cap \{x\}$", r is a set of literals). We denote the set of all possible worlds by $\mathcal{M}$.

For a set of sentences Γ of L, $[\Gamma]$ denotes the set of all possible worlds that entail Γ; i.e. $[\Gamma] = \{r \in \mathcal{M} : r \models \Gamma\}$. Often we use the notation $[x]$ for a sentence $x \in L$, as an abbreviation of $[\{x\}]$. For a theory K and a set of sentences Γ of L, we denote by $K + \Gamma$ the closure under $\models$ of $K \cup \Gamma$, i.e., $K + \Gamma = Cn(K \cup \Gamma)$. For a sentence $x \in L$ we often write $K + x$ as an abbreviation of $K + \{x\}$.

Finally, some definitions on binary relations. Let V be a nonempty set and R a binary relation in V. For any subset S of V, by $min(S, R)$ we denote the set $min(S, R) = \{w \in S : \text{for all } w' \in S, w'Rw \text{ entails } wRw'\}$. The elements in $min(S, R)$ are called *minimal in S* with respect to R (or simply *minimal in S*, when R is understood from the context). Observe that if R is irreflexive and anti-symmetric, the above definition of *min* is equivalent to: $min(S, R) = \{w \in S : \text{there is no } w' \in S, \text{ such that } w'Rw\}$.

We shall say that a binary R relation in V is a preorder iff R is *reflexive* and *transitive*. Moreover, R is said to be *total* iff for all $w, w' \in V$, wRw' or $w'Rw$.

The AGM Framework

Much research in belief change is based on the work of Alchourron, Gardenfors and Makinson (Alchourron, Gardenfors, and Makinson 1985), who have developed a research framework for this process, known as the *AGM framework*. In this section we shall briefly review the *rationality postulates* for the two most important types of belief change, namely *belief revision* and *belief contraction*, as well a constructive model for belief revision based on total preorders on possible worlds.

The AGM Postulates

In the AGM framework, belief revision is modeled as a function $*$ mapping a theory K and a sentence x to the theory $K * x$. Alchourron, Gardenfors, and Makinson have proposed the following set of postulates for belief revision:

(K*1) $K * x$ is a theory.

(K*2) $x \in K * x$.

(K*3) $K * x \subseteq K + x$.

(K*4) If $\neg x \notin K$ then $K + x \subseteq K * x$.

(K*5) If $x \not\models \bot$ then $K * x \not\models \bot$.

(K*6) If $x \equiv y$ then $K * x = K * y$.

(K*7) $K * (x \wedge y) \subseteq (K * x) + y$.

(K*8) If $\neg y \notin K * x$ then $(K * x) + y \subseteq K * (x \wedge y)$.

A revision function $*$ models the process by which a rational agents modifies her belief set K to incorporate some new information x. A *contraction function* $\div$ on the other hand models the process by which a rational agent uproots a belief x from K.

Formally, a contraction function is defined as a function $\div$ mapping a theory K and a sentence x to the theory $K \div x$, that satisfies the following postulates:

(K$\div$1) $K \div x$ is a theory.

(K$\div$2) $K \div x \subseteq K$.

(K$\div$3) If $x \notin K$ then $K \div x = K$.

(K$\div$4) If $\not\models x$ then $x \notin K \div x$.

(K$\div$5) If $x \in K$, then $K \subseteq (K \div x) + x$.

(K$\div$6) If $x \equiv y$ then $K \div x = K \div y$.

(K$\div$7) $(K \div x) \cap (K \div y) \subseteq K \div (x \wedge y)$.

(K$\div$8) If $x \notin K \div (x \wedge y)$ then $K \div (x \wedge y) \subseteq K \div x$.

Revision and contraction functions are connected through the following well known identities:

$$K \div x = (K * \neg x) \cap K \quad \text{(Harper Identity)}$$

$$K * x = (K \div \neg x) + x \quad \text{(Levi Identity)}$$

It has been shown that for every revision function $*$ satisfying (K*1) - (K*8), the function $\div$induced from $*$ via the Harper Identity, satisfies (K$\div$1) - (K$\div$8). And conversely, for every contraction function $\div$ satisfying (K$\div$1) - (K$\div$8), the function $*$ induced from $\div$ via the Levi Identity, satisfies (K*1) - (K*8) – see (Gardenfors 1988), (Peppas 2008), (Alchourron and Makinson 1982) for details.

Faithful Preorders

Apart from axiomatic approaches to belief change, a number of explicit constructions have also been proposed. One popular such construction is the one proposed by Grove, (Grove 1988), based on a structure called a *system of spheres*. Later Katsuno and Mendelzon, (Katsuno and Mendelzon 1991), re-casted Grove's model for the special case of finitary propositional logic, introducing the notion of *faithful preorders* on possible worlds.

Figure 1: Semiorder with Intransitive Indifference

For a theory K, a preorder on possible worlds $\preccurlyeq$ is said to be *faithful* to K iff it is total and such that the minimal worlds (wrt $\preccurlyeq$) are those satisfying K; i.e. $min(\mathcal{M}, \preccurlyeq) = [K]$.[4] Given a preorder $\preccurlyeq$ that is faithful to K, the revision of K by any sentence x can be defined as follows:

(RP) $[K * x] = min([x], \preccurlyeq)$.

Intuitively $\preccurlyeq$ represents a plausibility ranking on possible worlds: the more plausible a world r is, the lower it appears in the ranking. Hence (RP) essentially defines $K * x$ are the theory induced by the most plausible worlds satisfying the new information x.

Katsuno and Mendelzon proved that the family of functions induced from faithful preorders via (RP), are precisely those satisfying the AGM postulates for revision.

Semiorders

As mentioned in the introduction, semiorders were proposed in (Luce 1956) as a more natural alternative to total preorders for modelling preference.

Given a finite set of choices V, a semiorder $\prec$ in V is defined as a binary relation in V that satisfies the following axioms, for any $r_1, r_2, r_3, w \in V$:

(SO1) $r_1 \not\prec r_1$.

(SO2) If $r_1 \prec r_2 \prec r_3$ then $r_1 \prec w$ or $w \prec r_3$.

(SO3) If $r_1 \prec r_2$ and $r_3 \prec r_4$ then $r_1 \prec r_4$ or $r_3 \prec r_2$.

For any two choices $r_1, r_2 \in V$, we shall say that we are *indifferent* between r_1 and r_2, denoted $r_1 \sim r_2$, iff $r_1 \not\prec r_2$ and $r_2 \not\prec r_1$. It is not hard to verify that with semiorders, indifference is not in general transitive. Consider for example the semiorder $\prec$ depicted in Figure 1, where the arrows between alternatives indicate preference. It is easy to verify that $\prec$ satisfies (SO1) - (SO3). Moreover observe that $r_2 \sim r_3$, $r_3 \sim r_4$, and yet $r_2 \prec r_4$.

A central result on semiorders that sheds light to their inner workings, relates to their numerical representation. It was been shown, (Scott and Suppes 1958), (Rabinovitch 1977), that every semiorder $\prec$ in V can be mapped to a *utility function* $u : V \mapsto \mathbb{R}$ such that for all $r_1, r_2 \in V$, $r_1 \prec r_2$ iff $u(r_2) - u(r_1) \geqslant 1$, and $r_1 \sim r_2$ iff $|u(r_2) - u(r_1)| < 1$. Intuitively this result says that the agent differentiates between two alternatives r_1 and r_2 iff the difference in their

[4] To be precise, in (Katsuno and Mendelzon 1991) a belief state is represented as a sentence rather than as a theory, and accordingly, faithful preorders are associated to sentences rather than theories. However, given that we are working with a language built over finitely many propositional variables, the two approaches are equivalent.

corresponding utilities exceeds a certain threshold (set to 1 in this case). For example, the semiorder $\prec$ of Figure 1 can be represented by the following utility function u:

$$u(r_1) = 0$$
$$u(r_2) = 1$$
$$u(r_3) = 1.5$$
$$u(r_4) = 2$$

Semiorders in Belief Revision

As mentioned in the introduction, the main aim of this article is to re-build (the main elements of) the AGM framework using semiorders rather than total preorders as models of comparative plausibility. In this section we start with revision functions.

Let $\prec$ be a semiorder in the set $\mathcal{M}$ of possible worlds. The function $* : \mathcal{T} \times L \mapsto \mathcal{T}$ induced from $\prec$ is defined by condition (AS) below:[5]

(AS) $[K * x] = min([x], \prec)$.

We are interested in the class of functions induced from semiorders via (AS). Consider in particular the following set of postulates:

(A1) $K * x$ is a theory.

(A2) $x \in K * x$.

(A3) $K * x \subseteq K + x$.

(A4) If $\neg x \notin K$ then $K + x \subseteq K * x$.

(A5) If $x \not\models \bot$ then $K * x \not\models \bot$.

(A6) If $x \equiv y$ then $K * x = K * y$.

(A7) $K * (x \wedge y) \subseteq (K * x) + y$.

(A8) $K * (x \vee y) \subseteq (K * x) + (K * y)$.

(A9) If $K * y \not\subseteq (K * x) + y$ then $K * x \subseteq (K * y) + x$.

(A10) If $\neg y \in K * x$ and $\neg z \notin K * x$ then $K * (x \wedge y) \subseteq (K * z) + (x \wedge y)$.

Postulates (A1) - (A7) are identical to the AGM postulates (K*1) - (K*7) for revision. Postulate (K*8) however cannot simply be copied across.

To see this, assume that the semiorder $\prec$ in Figure 1 represents the comparative plausibility of the worlds r_1, r_2, r_3, r_4. Moreover assume that the agent's current belief set is $K = Cn(r_1)$. In (Gardenfors 1988) it was shown in that (K*8) is equivalent to the following condition:

(3.15) If $\neg y \notin K * (x \vee y)$ then $K * (x \vee y) \subseteq K * y$

Set $x = r_2 \vee r_3 \vee r_4$ and $y = r_3 \vee r_4$. From Figure 1 and (AS) it follows that $[K * x] = \{r_2, r_3\}$, $[K * y] = \{r_3, r_4\}$, and $[K*(x \vee y)] = \{r_2, r_3\}$. Hence $\neg y \notin K*(x \vee y)$ and therefore by (3.15), $[K * y] \subseteq [K * (x \vee y)]$. This of course leads us to a contradiction since $r_4 \in [K * y]$ and $r_4 \notin [K * (x \vee y)]$.

Postulate (K*8) is therefore too strong for semiorder-based revision functions. It turns out, as shown by Theorems 1, 2 below, that (A8) - (A10) are precisely the weakening of (K*8) required to characterise the functions induced from semiorders via (AS).

[5]Condition (AS) is derived from (RP) with semiorders $\prec$ replacing preorders $\preccurlyeq$.

We note that postulate (A8) is not new. It has been used, in a slightly different format, by Benferhat et. al. in their axiomatization of revision functions induced from *partial preorders* (see postulate (P7) in (Benferhat, Lagrue, and Papini 2005)).

Before we present our representation results we need to adjust the definition of faithfulness to semiorders.

Let $\prec$ be a semiorder in $\mathcal{M}$ and K a theory of L. We shall say that $\prec$ is *faithful to K* iff the following conditions are satisfied:

(i) If $r \in [K]$ then there is no $r' \in \mathcal{M}$ such that $r' \prec r$.

(ii) If $r \in [K]$ and $r' \in (\mathcal{M} - [K])$ then $r \prec r'$.

Theorem 1 *Let K be a theory and $*$ a revision function satisfying (A1) - (A10). Then there exists a semiorder $\prec$ faithful to K, that satisfies (AS).*

Proof. Define $\prec$ to be the following binary relation in $\mathcal{M}$: $r \prec r'$ iff $r' \notin [K * (r \vee r')]$. Firstly we show that $\prec$ satisfies (SO1). Consider any possible world $r \in \mathcal{M}$. By (A6), $[K * (r \vee r)] = [K * r]$. Moreover, by (A2), (A5), it follows that $[K * r] = \{r\}$, and therefore $[K * (r \vee r)] = \{r\}$. Hence $r \not\prec r$ as desired.

Next we show that (AS) is satisfied by showing $[K * x] \subseteq min([x], \prec)$ and $min([x], \prec) \subseteq [K * x]$.

Starting with the former, let r be any world in $[K * x]$. By (A2), $r \in [x]$. Consider now any world $r' \in [x]$. By (A7), $[K * x] \cap [r \vee r'] \subseteq [K * (x \wedge (r \vee r'))]$. Moreover notice that from $r, r' \in [x]$, it follows that $x \wedge (r \vee r') \equiv r \vee r'$. Hence by (A6), $[K * x] \cap [r \vee r'] \subseteq [K * (r \vee r')]$ and therefore $r \in [K * (r \vee r')]$. Hence $r' \not\prec r$. Since r' was chosen as an arbitrary element of $[x]$ it then follows that $r \in min([x], \prec)$. Hence $[K * x] \subseteq min([x], \prec)$.

We prove the converse by induction on the size of $[x]$. If $|[x]| = 1$, then $[x] = \{r\}$. In this case, from (A2) and (A5), and (SO1) (established above) we derive that $[K*x] = \{r\} = min([x], \prec)$.

Assume that $\min([x], \prec) \subseteq [K * x]$, for any $x \in L$ such that $|[x]| \leqslant k$ (Induction Hypothesis).

For the inductive step, assume $|[x]| = k + 1$, and let r be any world in $min([x], \prec)$. We will show that $r \in [K * x]$. To this end, consider any world $r' \in [x]$ that is different from r, and define y to be the sentence $y = x \wedge (\neg r')$. Clearly $[y] = [x] - \{r'\}$ and therefore $r \in min([y], \prec)$. Moreover, $|[y]| = k$, and therefore by the induction hypothesis, $r \in [K * y]$. Also notice that since $r \in min([x], \prec)$ and $r' \in [x]$, it follows that $r' \not\prec r$ and therefore, by the construction of $\prec$, $r \in [K * (r \vee r')]$. Hence by (A8), $r \in [K * (y \vee r \vee r')]$, and consequently by (A6), $r \in [K * x]$ as desired. Therefore (AS) holds.

Faithfulness of $\prec$ to K is straightforward. In particular, assume $r \in [K]$ and let r' be an arbitrary possible world. By (A3) - (A4), $[K * (r \vee r')] = [K] \cap [r \vee r']$ and therefore, $r \in [K * (r \vee r')]$. Hence $r' \not\prec r$. Notice moreover that if $r' \notin [K]$, then $[K * (r \vee r')] = [K] \cap [r \vee r'] = \{r\}$, and consequently $r \prec r'$. Hence $\prec$ is faithful to K.

To conclude the proof of Theorem 1 we need to show that $\prec$ also satisfies (SO2) and (SO3). To this end, we will first show that $\prec$ is transitive.

Consider any three worlds $r_1, r_2, r_3 \in \mathcal{M}$ such that $r_1 \prec r_2 \prec r_3$. By (AS) it follows that $[K * (r_1 \vee r_2 \vee r_3)] = \{r_1\}$.

Clearly then, $[K * (r_1 \vee r_2 \vee r_3)] \cap [r_2 \vee r_3] = \emptyset$ and $[K * (r_1 \vee r_2 \vee r_3)] \cap [r_1 \vee r_3] \neq \emptyset$, which by (A6) and (A10) entails that $[K*(r_1 \vee r_3)] \cap [r_2 \vee r_3] \subseteq [K*(r_2 \vee r_3)]$. Hence, since $r_2 \prec r_3$ we derive that $r_3 \notin [K * (r_1 \vee r_3)]$, and therefore $r_1 \prec r_3$. Consequently $\prec$ is transitive.

Next for (SO2), let $r_1, r_2, r_3 \in \mathcal{M}$ be any three worlds such that $r_1 \prec r_2 \prec r_3$. Assume towards contradiction that there is a $w \in \mathcal{M}$ such that $r_1 \not\prec w$ and $w \not\prec r_3$. Notice that if $r_2 \prec w$ or $r_3 \prec w$, then transitivity gives us $r_1 \prec w$ which of course contradicts our assumption. Hence $r_2 \not\prec w$ and $r_3 \not\prec w$ (on top of $r_1 \not\prec w$), which combined with (SO1) and (AS) gives us, $w \in [K * (r_1 \vee r_2 \vee r_3 \vee w)]$ and $r_2, r_3 \notin [K * (r_1 \vee r_2 \vee r_3 \vee w)]$. Hence, $[K * (r_1 \vee r_2 \vee r_3 \vee w)] \cap [r_2 \vee r_3] = \emptyset$ and $[K * (r_1 \vee r_2 \vee r_3 \vee w)] \cap [r_3 \vee w] \neq \emptyset$, which by (A6) and (A10) entails $[K * (r_3 \vee w)] \cap [r_2 \vee r_3] \subseteq [K * (r_2 \vee r_3)]$. Consequently, since $r_3 \notin [K * (r_2 \vee r_3)]$ it follows that $r_3 \notin [K * (r_3 \vee w)]$, and therefore $w \prec r_3$, which of course contradicts our initial assumption.

Finally for (SO3), assume towards contradiction that for some $r_1, r_2, r_3, r_4 \in \mathcal{M}$, $r_1 \prec r_2, r_3 \prec r_4, r_1 \not\prec r_4$, and $r_3 \not\prec r_2$. From transitivity we then derive that $r_2 \not\prec r_4$, and $r_4 \not\prec r_2$. Consequently, by (SO1) and (AS), $r_4 \in [K * (r_1 \vee r_2 \vee r_4)]$ and $r_4 \notin [K * (r_2 \vee r_3 \vee r_4)]$. (A9) then entails that $[K * (r_2 \vee r_3 \vee r_4)] \cap [r_1 \vee r_2 \vee r_4] \subseteq [K * (r_1 \vee r_2 \vee r_4)]$. On the other hand, from (SO1) and (AS) we derive that $r_2 \in [K * (r_2 \vee r_3 \vee r_4)]$, and therefore $r_2 \in [K * (r_1 \vee r_2 \vee r_4)]$. This of course contradicts $r_1 \prec r_2$. ■

In proving the converse of Theorem 1, the following result will be useful:

Lemma 1 *Let $\prec$ be a semiorder in $\mathcal{M}$ and $S \subseteq \mathcal{M}$ a nonempty set of worlds. Then,*

(a) *$\prec$ is transitive.*

(b) *$min(S, \prec) \neq \emptyset$.*

(c) *if $min(S, \prec) \neq \emptyset$, $r \in S$, and $r \notin min(S, \prec)$, then there exists a $r' \in min(S, \prec)$ such that $r' \prec r$.*

Proof. The transitivity of a semiorder follows directly from (Luce 1956). Nevertheless we include the proof herein for completeness. Consider any three worlds $r_1, r_2, r_3 \in \mathcal{M}$ such that $r_1 \prec r_2$ and $r_2 \prec r_3$. From (SO3) it follows that $r_1 \prec r_3$ or $r_2 \prec r_2$. The latter is excluded by (SO1), and consequently $r_1 \prec r_3$ as desired.

Next we prove the third statement of Lemma 1 by induction on the size of S. If $|S| = 1$, then $S = \{r\}$ for some world r. Consequently by (SO1), $min(S, \prec) = \{r\}$ and hence (c) trivially holds.

Assume that (c) is true for all nonempty sets with cardinality no greater than k (Induction Hypothesis).

For the inductive step, consider a set $S \subseteq \mathcal{M}$ such that $|S| = k + 1$, and assume that $min(S, \prec) \neq \emptyset$, and for some $r \in S$, $r \notin min(S, \prec)$. Then there is a $r' \in S$ such that $r' \prec r$. From transitivity and (SO1) we then derive that $r' \neq r$ and $r \not\prec r'$. Consider now the set $S - \{r\}$. Clearly, $r' \in (S - \{r\})$ and $|S - \{r\}| = k$. If $r' \in min(S - \{r\}, \prec)$, then from $r \not\prec r'$ it follows that $r' \in min(S, \prec)$, and therefore $r' \prec r$ gives us (c). Assume therefore that $r' \notin min(S - \{r\}, \prec)$. Then by the induction hypothesis, there exists a $r'' \in min(S - \{r\}, \prec)$ such that $r'' \prec r'$. Then transitivity entails $r'' \prec r$. Moreover (SO1) together with transitivity entail $r \not\prec r''$. Hence $r'' \in min(S, \prec)$. Combined with $r'' \prec r$ we derive (c).

Finally we prove (b), once again using induction on the size of S. If $|S| = 1$, then $S = \{r\}$ for some world r. Then by (SO1), $min(S, \prec) = \{r\}$ and hence $min(S, \prec) \neq \emptyset$.

Assume that (b) is true for all nonempty sets $S \subseteq \mathcal{M}$ of size up to k (Induction Hypothesis).

Suppose now that $S \subseteq \mathcal{M}$ is a set with cardinality $|S| = k + 1$. Pick an arbitrary world $r \in S$. If $r \in min(S, \prec)$ then (b) is satisfied. Assume therefore that $r \notin min(S, \prec)$. Then there is a $r' \in S$ such that $r' \prec r$. From (SO1) and transitivity it follows that $r' \neq r$ and $r \not\prec r'$. Hence $r' \in (S - \{r\})$. If $r' \in min(S - \{r\}, \prec)$, from $r \not\prec r'$ we derive that $r' \in min(S - \{r\}, \prec)$ and therefore $min(S, \prec) \neq \emptyset$ as desired. Assume therefore that $r' \notin min(S - \{r\}, \prec)$. By the induction hypothesis we derive that $min(S - \{r\}, \prec) \neq \emptyset$. Hence from (c) it follows that there is a $r'' \in min(S - \{r\}, \prec)$ such that $r'' \prec r'$. Transitivity then entails that $r'' \prec r$, and consequently from (SO1) and transitivity it follows that $r \not\prec r''$. This again entails that $r'' \in min(S, \prec)$, and therefore, $min(S, \prec) \neq \emptyset$. ■

With the aid of Lemma 1 we can now prove the soundness of our postulates wrt the semiorder-based construction:

Theorem 2 *Let K be a theory and $\prec$ a semiorder faithful to K. The revision function $*$ induced from $\prec$ via (AS), satisfies (A1) - (A10).*

Proof. The validity of (A1), (A2), and (A6) is straightforward. For (A3), notice that if $\neg x \in K$ then $K + x = L$, and therefore $K * x \subseteq K + x$. Assume now that $\neg x \notin K$, or equivalently, $[K] \cap [x] \neq \emptyset$. Then by the faithfulness of $\prec$ to K it follows that $min([x], \prec) = [K] \cap [x]$. Hence by (AS), $K * x = K + x$. This proves (A3). Moreover, the second part of the argument also proves (A4).

For (A5), assume that $x \not\models \bot$ or equivalently that $[x] \neq \emptyset$. Then from Lemma 1.(b) it follows that $min([x], \prec) \neq \emptyset$. Hence $K * x$ is consistent as desired.

For (A7), let r be any world in $[(K * x) + y]$ and assume towards contradiction that $r \notin [K * (x \wedge y)]$. From $r \in [(K * x) + y]$ it follows that $[x \wedge y] \neq \emptyset$, and hence from $r \notin [K * (x \wedge y)]$ we derive that for some $r' \in [x] \cap [y]$, $r' \prec r$. Consequently, $r \notin min([x], \prec)$, which of course contradicts $r \in [(K * x) + y]$.

For (A8), assume that $r \in [K * x] \cap [K * y]$. Clearly then $[x] \neq \emptyset$ and $[y] \neq \emptyset$. Suppose contrary to (A8), that $r \notin [K * (x \vee y)]$. Then, since $r \in [x \vee y]$ we derive that there is a $r' \in [x \vee y]$ such that $r' \prec r$. Since $[x \vee y] = [x] \cup [y]$ it follows that $r \in [x]$ or $r \in [y]$. In the first case, from $r' \prec r$ we derive that $r \in min([x], \prec)$. In the second case we derive that $r \in min([y], \prec)$. Hence in both cases we have that $r \notin [K * x] \cap [K * y]$, contradicting our initial assumption.

For (A9), assume that for some $x, y \in L$, $[K * x] \cap [y] \nsubseteq [K * y]$. Then there is a $r \in [K * x] \cap [y]$ such that $r \notin [K * y]$. From $r \in [K * x] \cap [y]$ it follows that $[x] \neq \emptyset$ and $[y] \neq \emptyset$. Hence from $r \notin [K * y]$ we derive that there is a $r' \in [y]$ such that $r' \prec r$. Now assume toward contradiction that $[K * y] \cap [x] \nsubseteq [K * x]$, and let $w \in \mathcal{M}$ be such that $w \in [K * y] \cap [x]$ and $w \notin [K * x]$. From the latter we derive that there is a $w' \in [x]$ such that $w' \prec w$. Then (SO3) entails that $r' \prec w$ or $w' \prec r$. In the first case, since $r' \in [y]$, it follows that $w \notin min([y], \prec)$, which of course contradicts $w \in [K * y]$. In the second case, from $w' \in [x]$ we derive that $r \notin min([x], \prec)$, which contradicts $r \in [K * x]$. Hence, in both case we derive a contradiction, and therefore (A9) is true.

Finally for (A10), let $x, y, z \in L$ be such that $[K*x] \cap [y] = \emptyset$, and $[K * x] \cap [z] \neq \emptyset$. Assume towards contradiction that there exists a $r \in [K*z] \cap [x] \cap [y]$ such that $r \notin [K*(x \wedge y)]$. Then, by Lemma 1.(c), there exists a $r' \in min([x \wedge y], \prec)$ such that $r' \prec r$. From $[K * x] \cap [y] = \emptyset$ we then derive that $r' \notin min([x], \prec)$. Hence from Lemma 1.(c) it follows that there is a $r'' \in min([x], \prec)$ such that $r'' \prec r'$. Moreover from $[K * x] \cap [z] \neq \emptyset$ we derive that there is a $w \in [z]$ such that $w \in [K * x]$. From $r'' \prec r' \prec r$ and (SO2) we derive that $r'' \prec w$ or $w \prec r$. In the first case, since $r'' \in [x]$, it follows that $w \notin min([x], \prec)$, contradicting $w \in [K * x]$. In the latter case, since $w \in [z]$, we derive that $r \notin min([z], \prec)$, contradicting $r \in [K * z]$. Hence in both cases we derive a contradiction. Thus (A10) hold. ■

Semiorders in Belief Contraction

In this section we identify the class of contraction functions that correspond, via the Harper and Levi Identities, to semiorder-induced revision functions.

In particular, consider the following postulates:

(B1) $K \div x$ is a theory.

(B2) $K \div x \subseteq K$.

(B3) If $x \notin K$ then $K \div x = K$.

(B4) If $\not\models x$ then $x \notin K \div x$.

(B5) If $x \in K$ then $K \subseteq (K \div x) + x$.

(B6) If $x \equiv y$ then $K \div x = K \div y$.

(B7) $(K \div x) \cap (K \div y) \subseteq K \div (x \wedge y)$.

(B8) $K \div (x \wedge y) \subseteq (K \div x) + (K \div y)$.

(B9) If $K \div y \not\subseteq (K \div x) + \neg y$ then $K \div x \subseteq (K \div y) + \neg x$.

(B10) If $x \vee y \in K \div x$, and $x \vee z \notin K \div x$ then $K \div (x \vee y) \subseteq K \div (x \vee z) + \neg(x \vee y)$.

Postulates (B1) - (B7) are identical to (K÷1) - (K÷7). On the other hand (B8) - (B10) are weaker variants of (K÷8) (assuming the presence of (B1) - (B7)).

The following results show that the class of contraction functions satisfying (B1) - (B10) corresponds precisely, via the Levi and Harper Identities, to the class of revision functions satisfying (A1) - (A10).

Theorem 3 *Let K be a theory, $*$ a revision function satisfying (A1) - (A10), and $\div$ the function induced from $*$ via the Harper Identity. Then $\div$ satisfies (B1) - (B10).*

Proof. From Theorems 3.4, 3.5 in (Gardenfors 1988), it follows that $\div$ satisfies (B1) - (B7).

For (B8), consider any $x, y \in L$. By the Harper Identity, $[K \div x] \cap [K \div y] = ([K * \neg x] \cup [K]) \cap ([K * \neg y] \cup [K]) = ([K*\neg x] \cap [K*\neg y]) \cup ([K*\neg x] \cap [K]) \cup ([K*\neg y] \cap [K]) \cup [K]$. Moreover by (B2), $[K] \subseteq [K * \neg x]$, $[K] \subseteq [K * \neg y]$, and by (A6) and (A8), $[K*\neg x] \cap [K*\neg y] \subseteq [K*\neg(x \wedge y)]$. Consequently, $[K \div x] \cap [K \div y] \subseteq [K * \neg(x \wedge y)] \cup [K]$. Hence, by the Harper Identity, $K \div (x \wedge y) \subseteq (K \div x) + (K \div y)$ as desired.

For (B9), consider any $x, y \in L$ such that $[K \div x] \cap [\neg y] \not\subseteq [K \div y]$. Then by the Harper Identity, $([K * \neg x] \cup [K]) \cap [\neg y] \not\subseteq [K*\neg y] \cup [K]$ and consequently, $([K*\neg x] \cap [\neg y]) \cup ([K] \cap [\neg y]) \not\subseteq [K * \neg y] \cup [K]$. Since by (A3) $[K] \cap [\neg y] \subseteq [K * \neg y]$, we then derive that $[K * \neg x] \cap [\neg y] \not\subseteq [K * \neg y]$. Then by (A9) it follows that $[K * \neg y] \cap [\neg x] \subseteq [K * \neg x]$. Hence $([K*\neg y] \cap [\neg x]) \cup [K] \subseteq [K*\neg x] \cup [K]$ and therefore $([K * \neg y] \cup [K]) \cap ([\neg x] \cup [K]) \subseteq [K * \neg x] \cup [K]$. By the Harper Identity we then derive $([K \div y] \cap ([\neg x] \cup [K]) \subseteq [K \div x]$, and therefore, $([K \div y] \cap [\neg x]) \cup ([K \div y] \cap [K]) \subseteq [K \div x]$. Hence $[K \div y] \cap [\neg x] \subseteq [K \div x]$ or equivalently, $K \div x \subseteq (K \div y) + \neg x$.

Finally for (B10), assume that for some $x, y, z \in L$, $[K \div x] \cap [\neg x] \subseteq [y]$, and $[K \div x] \cap [\neg x] \not\subseteq [z]$. Firstly assume that $x \notin K$. Then by (B3), $K \div x = K$. Hence from $[K \div x] \cap [\neg x] \not\subseteq [z]$ we derive that $(x \vee z) \notin K$, and consequently by (B3), $K \div (x \vee z) = K$. Then by (B2), $K \div (x \vee y) \subseteq K \div (x \vee z)$, and therefore, $K \div (x \vee y) \subseteq K \div (x \vee z) + \neg(x \vee y)$ as desired.

Assume therefore that $x \in K$. From $[K \div x] \cap [\neg x] \subseteq [y]$, and $[K \div x] \cap [\neg x] \not\subseteq [z]$ and the Harper Identity it follows that $([K * \neg x] \cup [K]) \cap [\neg x] \subseteq [y]$, and $([K * \neg x] \cup [K]) \cap [\neg x] \not\subseteq [z]$. Consequently, $([K * \neg x] \cap [\neg x]) \cup ([K] \cap [\neg x]) \subseteq [y]$, and $([K * \neg x] \cap [\neg x]) \cup ([K] \cap [\neg x]) \not\subseteq [z]$. Since $x \in K$ we then derive, $[K*\neg x] \cap [\neg x] \cap [\neg y] = \emptyset$, and $[K*\neg x] \cap [\neg x] \cap [\neg z] \neq \emptyset$, or equivalently, $[K*\neg x] \cap [\neg x \wedge \neg y] = \emptyset$, and $[K*\neg x] \cap [\neg x \wedge \neg z] \neq \emptyset$. Axiom (A10) then entails that $[K*(\neg x \wedge \neg z)] \cap [\neg x \wedge \neg y] \subseteq [K*(\neg x \wedge \neg y)]$. Hence $([K*(\neg x \wedge \neg z)] \cap [\neg x \wedge \neg y]) \cup [K] \subseteq [K*(\neg x \wedge \neg y)] \cup [K]$, and consequently, by (A6), $([K * \neg(x \vee z)] \cup [K]) \cap ([\neg x \wedge \neg y] \cup [K]) \subseteq [K*\neg(x \vee y)] \cup [K]$. Therefore by the Harper Identity, $[K \div (x \vee z)] \cap ([\neg(x \vee y)] \cup [K]) \subseteq [K \div (x \vee y)]$. Hence $([K \div (x \vee z)] \cap [\neg(x \vee y)]) \cup ([K \div (x \vee z)] \cap [K]) \subseteq [K \div (x \vee y)]$, and consequently, $K \div (x \vee y) \subseteq K \div (x \vee z) + \neg(x \vee y)$ as desired. ■

Theorem 4 *Let K be a theory, $*$ a contraction function satisfying (B1) - (B10), and $*$ the function induced from $*$ via the Levi Identity. Then $*$ satisfies (A1) - (A10).*

Proof. From Theorems 3.2, 3.3 in (Gardenfors 1988), it follows that $*$ satisfies (A1) - (A7).

For (A8), let x, y be any sentences in L. By the Levi Identity, $[K * x] \cap [K * y] = [K \div \neg x] \cap [x] \cap [K \div \neg y] \cap [y]$ and therefore by (B8) and (B6), $[K * x] \cap [K * y] \subseteq [K \div \neg(x \vee y)] \cap [x \wedge y]$. Moreover notice that $[x \wedge y] \subseteq [x \vee y]$ and consequently we derive $[K * x] \cap [K * y] \subseteq [K \div \neg(x \vee y)] \cap [x \vee y]$. Hence from the Levi Identity it follows that $K * (x \vee y) \subseteq (K * x) + (K * y)$ as desired.

For (A9), assume that for some $x, y \in L$, $[K * x] \cap [y] \not\subseteq [K * y]$. Then by the Levi Identity we have that $[K \div \neg x] \cap [x] \cap [y] \not\subseteq [K \div \neg y] \cap [y]$ and consequently $[K \div \neg x] \cap [y] \not\subseteq [K \div \neg y]$. Then by (B9), $[K \div \neg y] \cap [x] \subseteq [K \div \neg x]$, and therefore $[K \div \neg y] \cap [x] \subseteq [K \div \neg x] \cap [x]$. This again entails $[K \div \neg y] \cap [x] \cap [y] \subseteq [K \div \neg x] \cap [x]$ which by the Levi Identity entails $[K * y] \cap [x] \subseteq [K * x]$, or equivalently, $K * x \subseteq (K * y) + x$ as desired.

Finally for (A10), assume that for some $x, y, z \in L$, $[K * x] \cap [y] = \emptyset$ and $[K * x] \cap [z] \neq \emptyset$. Then by the Levi Identity we get, $[K \div \neg x] \cap [x] \cap [y] = \emptyset$ and $[K \div \neg x] \cap [x] \cap [z] \neq \emptyset$, or equivalently, $\neg y \in (K \div \neg x) + x$ and $\neg z \notin (K \div \neg x) + x$. Hence by (B10) and (B6), $[K \div \neg(x \wedge z)] \cap [x \wedge y] \subseteq [K \div \neg(x \wedge y)]$, and consequently, $[K \div \neg(x \wedge z)] \cap [x \wedge y] \subseteq [K \div \neg(x \wedge y)] \cap [x \wedge y]$. This again entails, $[K \div \neg(x \wedge z)] \cap [x \wedge z] \cap [x \wedge y] \subseteq [K \div \neg(x \wedge y)] \cap [x \wedge y]$, and therefore by the Levi Identity we derive, $K * (x \wedge y) \subseteq K * (x \wedge z) + (x \wedge y)$. Moreover by (A6) - (A7), $K * (x \wedge z) \subseteq (K * z) + x$ and therefore $K * (x \wedge y) \subseteq (K * z) + (x \wedge y)$ as desired. ■

Related Work

To our knowledge, this is the first time that semiorders are used in Belief Change. Yet a number of researchers have considered other weaker alternatives of total preorders as models of comparative plausibility. We have already mentioned (Benferhat, Lagrue, and Papini 2005), in which the authors study revision functions based on *partial preorders*.

Another instance is (Rott to appear). In this paper Rott, focuses on contraction functions, and uses the metaphor of a multilayer edifice to classify them. In particular, on the ground floor of the edifice one encounters the contraction functions satisfying the basic postulates $(K \div 1)$ - $(K \div 6)$. On the top floor dwell the fully fledged AGM contraction functions satisfying all eight postulates $(K \div 1)$ - $(K \div 8)$. In the space between the ground and the top floors, Rott identifies three more floors, each of which is of interest in its own right. The 3rd floor of the edifice is the one most relevant to the present work: on this floor live the relational partial meet contraction functions induced from *interval orders* (on remainders).[6] We recall that an interval order is a binary relation $<$ satisfying (SO1) and (SO3); hence every semiorder is an interval order, but not vice versa.

Rott proves that the functions residing on the 3rd floor can be characterised axiomatically by $(K \div 1)$ - $(K \div 7)$, together with the postulates $(K \div 8c)$ and $(K \div 8d)$ below; notice the close resemblance between $(K \div 8d)$ and our postulate (B8):

$(K \div 8c)$ If $y \in K \div (x \wedge y)$, then $K \div (x \wedge y) \subseteq K \div x$.

$(K \div 8d)$ $K \div (x \wedge y) \subseteq (K \div x) \cup (K \div y)$.

In view of the prominence of semiorders in modeling intransitivity of preference indifference (and of plausibility indifference), we believe that an extra floor could be added to Rott's edifice between the 3rd and the top floor, to accommodate the contraction functions of the previous section.

Interval orders have also been used in (Booth and Meyer 2011). However, whereas herein we use semiorders to model comparative plausibility, in (Booth and Meyer 2011) interval orders are used at a meta-level to guide the *iterated revision* process (with comparative plausibility still been modelled with total preorders like in the classical AGM framework).

Conclusion

Researchers in the *preference modelling* community have long argued in favour of *semiorders* over *total preorders* as a more natural model for preference. The main advantage of semiorders is the intransitivity of indifference.

Herein we have argued that in the context of Belief Change indifference in plausibility could also be non-transitive. We therefore re-built the AGM framework with semiorders replacing total preorders. In particular we characterize axiomatically the class of revision functions induced by semiorders, along with the corresponding class or contraction functions (via the Levi and Harper Identities). Future work will include similar results for the epistemic entrenchment and the partial meet models.

Of course Belief Change is not the only area in Knowledge Representation that uses plausibility rankings. *Nonmonotonic Reasoning* for example also relies heavily on such rankings. Moreover, like in Belief Change, plausibility indifference in Nonmonotonic Reasoning is also typically assumed to be transitive. Hence we believe that the study of semiorders in the context of Nonmonotonic Reasoning is an important avenue for future work that could lead to significant contributions.

[6] See (Gardenfors 1988), (Peppas 2008) for details on the partial meet model for contraction.

Acknowledgements

We are grateful to Abhaya Nayak and Maurice Pagnucco for many valuable discussion on this work. Moreover we would like to thank the anonymous reviewers for their detailed comments and insightful input, as well as Hans Rott for bringing (Rott to appear) to our attention.

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