Strategyproof Mechanisms for One-Dimensional Hybrid and Obnoxious Facility Location Models

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Abstract

We consider a strategic variant of the facility location problem. We would like to locate a facility on a closed interval. There are n agents located on that interval, divided into two types: type 1 agents, who wish for the facility to be as far from them as possible, and type 2 agents, who wish for the facility to be as close to them as possible. Our goal is to maximize a form of aggregated social benefit: maxisum– the sum of the agents' utilities, or the egalitarian objective– the minimal agent utility. The strategic aspect of the problem is that the agents' locations are not known to us, but rather reported to us by the agents– an agent might misreport his location in an attempt to move the facility away from or towards to his true location. We therefore require the facility-locating mechanism to be strategyproof, namely that reporting truthfully is a dominant strategy for each agent. As simply maximizing the social benefit is generally not strategyproof, our goal is to design strategyproof mechanisms with good approximation ratios.

In this paper, we provide a best-possible 3- approximate deterministic strategyproof mechanism, as well as a $\frac{23}{13}$- approximate randomized strategyproof mechanism, both for the maxisum objective. We provide lower bounds of 3 and $\frac{3}{2}$ on the approximation ratio attainable for maxisum, in the deterministic and randomized settings, respectively. For the egalitarian objective, we show that no bounded approximation ratio is attainable in the deterministic setting, and provide a lower bound of $\frac{3}{2}$ for the randomized setting. To obtain our deterministic lower bounds, we characterize all deterministic strategyproof mechanisms when all agents are of type 1. Finally, while still restricting ourselves to agents of type 1 only, we consider a generalized model that allows an agent to control more than one location. In this generalized model, we provide best-possible 3- and $\frac{3}{2}$- approximate strategyproof mechanisms for the maxisum objective in the deterministic and randomized settings, respectively.

1 Introduction

Consider the problem of locating a single facility on a closed interval. There are n agents, located in the interval, divided into two types: type 1 agents, who wish for the facility to be as far away from them as possible, and type 2 agents, who wish for the facility to be as close to them as possible. In particular, the utility of a type 1 agent equals his distance from the facility, while the utility of a type 2 agent equals the length of the interval minus his distance from the facility. A social planner wishes to locate the facility in a way that maximizes some aggregated measure of the agents' utilities. However, we are interested in a variant of the problem first introduced in (Procaccia and Tennenholtz 2013), in which the locations of the agents are not known to the planner, but rather are reported to the planner by the agents themselves. In that case, the agents may misreport their locations if doing so will cause the planner to locate the facility in a location more desirable to them. Due to this strategic aspect, the planner cannot simply locate the facility at the optimal location with respect to the reports. Instead, we require the mechanism used by the planner to be *strategyproof*: truthfully reporting his location is a dominant strategy for each agent. Subject to this requirement, the planner's goal is to optimize the social benefit, in terms of worst case approximation ratio. We consider maximizing two social benefit functions: the maxisum function, which is simply the sum of the agents' utilities, and the egalitarian function, which is the minimum agent utility.

The strategic facility location problem and its variations have received a lot of attention in the recent literature. The case of the unbounded interval with type 2 agents alone was studied in (Procaccia and Tennenholtz 2013), (Feldman and Wilf 2013) and (Feigenbaum, Sethuraman, and Ye 2013); a notable characterization of deterministic strategyproof mechanisms in this setting is given in (Moulin 1980). The case of the bounded interval with agents of type 1 alone, called the *obnoxious facility location* problem, was introduced in (Cheng, Yu, and Zhang 2013) and further explored in (Ibara and Nagamochi 2012). There is much related research, considering different graph topologies, different number of facilities, and more: see, for example, (Cheng et al. 2013), (Alon et al. 2010), and (Alon et al. 2009). To the best of our knowledge, our paper is the first to consider the generalized, hybrid model which contains both types of agents.

Our main findings are summarized below:

- We design a 3- approximate deterministic strategyproof mechanism, and a $\frac{23}{13}$- approximate randomized strategyproof mechanism for the maxisum objective.

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- We characterize deterministic strategyproof mechanisms when only type 1 agents are present.
- We prove a lower bound of 3 on the approximation ratio of deterministic strategyproof mechanisms for the maxisum objective, thus proving the optimality of the mechanism we provide for this setting. We also show that no deterministic strategyproof mechanism can provide a bounded approximation ratio for the egalitarian objective. These bounds hold even when all agents are of type 1.
- We prove a lower bound of $\frac{3}{2}$ on the approximation ratio of randomized strategyproof mechanisms for both the maxisum and egalitarian objectives. These bounds hold even when all agents are of type 1.
- We consider a generalized model that allows an agent to control more than one location. In this model, we provide a 3- and $\frac{3}{2}$- approximate strategyproof mechanisms for the deterministic and randomized settings respectively, assuming only type 1 agents are present (matching our proven lower bounds).

In the interest of space, complete proofs of theorems are included in the extended version of the paper, which can be found online at www.columbia.edu/ js1353/hybrid.pdf.

2 Model

Let $N = \{1, 2, \dots, n\}$ be the set of agents, and let I be the closed interval. We assume, without loss of generality, that $I = [0, 2]$. Each agent $i \in N$ reports a location $x_i \in I$. The vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is a location profile; for any $\alpha \in I$, we also use the notation $(\alpha, \mathbf{x_{-i}}) = (x_1, x_2, \dots, x_{i-1}, \alpha, x_{i+1}, \dots, x_n)$, where $\mathbf{x_{-i}} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ is a partial location profile of all agents but i. A *deterministic* mechanism is a collection of functions $f = \{f_n | n \in \mathbb{N}\}$ such that each $f_n : I^n \to I$ maps each location profile $\mathbf{x} = (x_1, x_2, \dots, x_n)$ to the location of the facility. We use $f(\mathbf{x})$ instead of $f_n(\mathbf{x})$ when n is clear from the context. Similarly, a *randomized* mechanism is a collection of functions f that maps each location profile to a probability distribution over I: if $f(\mathbf{x})$ is the distribution π, then the facility is located by drawing a single sample from π.

We study deterministic and randomized mechanisms for the problem of locating a single facility when the location of any agent is *private* information to that agent and cannot be observed or otherwise verified. It is therefore critical that the mechanism be *strategyproof*—it should be optimal for each agent i to report his *true* location x_i. To make this precise, we assume that if the facility is located at y, an agent's utility, equivalently benefit, is either $B_i(x_i, y) = |x_i - y|$, if he is a type 1 agent, or $B_i(x_i, y) = 2 - |x_i - y|$, if he is a type 2 agent[1]. If the location of the facility is randomly distributed with distribution π, then the benefit of agent i is simply $\mathbb{E}_{Y\sim\pi}[B_i(x_i, Y)]$, where Y is a random variable with distribution π. We denote the set of type j agents as N_j for $j = 1, 2$. The formal definition of strategyproofness is now:

Definition 1. *A deterministic mechanism f is strategyproof if for each $i \in N$, each $x_i, x'_i \in I$, and for each $\mathbf{x_{-i}} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in I^{n-1}$, $B_i(x_i, f(x_i, \mathbf{x_{-i}})) \geq B_i(x_i, f(x'_i, \mathbf{x_{-i}}))$; a randomized mechanism f is strategyproof if $E_{Y\sim f(x_i,\mathbf{x_{-i}})}[B_i(x_i, Y)] \geq E_{Y\sim f(x'_i,\mathbf{x_{-i}})}[B_i(x_i, Y)]$.*

In this paper we assume that locating a facility at y when the location profile is $\mathbf{x} = (x_1, x_2, \dots, x_n)$ gives the *social benefit* $sb(\mathbf{x}, y)$, where we consider two possible options for sb: *maxisum*, defined by $sb(\mathbf{x}, y) = \sum_{i=1}^{n} B_i(x_i, y)$, and *egalitarian*, $sb(\mathbf{x}, y) = \min_{i\in N} B_i(x_i, y)$. When the facility is located according to a probability distribution π, *maxisum* is defined as $sb(\mathbf{x}, \pi) = E_{Y\sim\pi}[\sum_{i=1}^{n} B_i(x_i, Y)]$, and *egalitarian* as $sb(\mathbf{x}, \pi) = E_{Y\sim\pi}[\min_{i\in N} B_i(x_i, Y)]$. The goal is to find a strategyproof mechanism that does well with respect to maximizing (either definition of) the social benefit. A natural mechanism is the "optimal" mechanism: each location profile $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is mapped to $OPT(\mathbf{x})$, defined as[2] $OPT(\mathbf{x}) \in \arg\max_{y\in I} sb(\mathbf{x}, y)$. However, the optimal mechanism is not generally strategyproof. Given that strategyproofness and optimality cannot be achieved simultaneously, it is necessary to find a tradeoff. In this paper we shall restrict ourselves to strategyproof mechanisms that approximate the optimal social benefit as best as possible: an α- approximation ($\alpha \in [1, \infty)$) algorithm guarantees at least a $\frac{1}{\alpha}$ fraction of the optimal social benefit for every instance of the problem. Formally, the approximation ratio of an algorithm A is $\sup_T\{OPT(T)/A(T)\}$, where the supremum is taken over all possible instances T of the problem; and $A(T)$ and $OPT(T)$ are, respectively, the benefits obtained by algorithm A and the optimal algorithm on the instance T. Our goal is to design strategyproof mechanisms whose approximation ratio is as close to 1 as possible.

3 Deterministic and Randomized Mechanisms for the Hybrid Model

In this section, we provide a best-possible 3- approximate deterministic strategyproof mechanism for the maxisum objective, as well as a $\frac{23}{13}$- approximate randomized strategyproof mechanism for the same objective.

Theorem 1. *Let $R = \{i : i \in N_1, x_i \leq 1\} \cup \{i : i \in N_2, x_i \geq 1\}$, $L = \{i : i \in N_1, x_i > 1\} \cup \{i : i \in N_2, x_i < 1\}$. Let f be the mechanism that locates the facility at 2 if $|R| \geq |L|$ and at 0 otherwise. Then f is a 3- approximate strategyproof mechanism for the maxisum objective.*

Proof sketch. Strategyproofness is easy. For the approximation ratio, let $\mathbf{x}$ be a location profile. Assume without loss of generality that $f(\mathbf{x}) = 2$, and let R_j and L_j be the sets of agents of type j in R and L, respectively. To show that $sb(\mathbf{x}, a) \leq 3sb(\mathbf{x}, f(\mathbf{x}))$ for every $a \in I$, we first show that the worst case approximation ratio occurs when the

[1]The number 2 is chosen merely because it is the length of I. With this choice, the utility of each agent is between 0 and 2 regardless of his type.

[2]If the social benefit at $\mathbf{x}$ is maximized by multiple locations y, an exogenous tie-breaking rule is used to select one of the optimal locations.

profile is $x_i = 1$ for $i \in R_1 \cup R_2$, $x_i = 0$ for $i \in L_2$, and $x_i = 2$ for $i \in L_1$; this latter observation follows from noting that that the approximation ratio must increase as each agent moves toward his location in this profile. We further show that the worst case occurs when the facility is located at $a \in \{0, 1\}$. The upper bound of 3 is established by analyzing the two resulting cases.

Later, we prove a lower bound of 3 on the approximation ratio under strategyproofness. Thus, the approximation ratio achieved by this mechanism is best-possible. Moreover, in the obnoxious facility model ($N_2 = \emptyset$), the above mechanism reduces to the deterministic mechanism proposed in (Cheng, Yu, and Zhang 2013), where it is proved that this mechanism a 3- approximation for that special case.

We now use randomization in an attempt to improve the approximation ratio. Getting a 2- approximation is easy: choosing each endpoint with probability $\frac{1}{2}$ is a 2- approximate strategyproof mechanism [3]. However, we can do better:

Theorem 2. *Let $p_1 = \frac{12}{23}$, $p_2 = \frac{8}{23}$, and $p_3 = \frac{3}{23}$. Consider the following randomized mechanism f. If $|R| \geq |L|$, then $P(f(\mathbf{x}) = 2) = p_1$ and $P(f(\mathbf{x}) = 0) = p_2$; if $|R| < |L|$, then $P(f(\mathbf{x}) = 2) = p_2$ and $P(f(\mathbf{x}) = 0) = p_1$; and either way, $P(f(\mathbf{x}) = 1) = p_3$. The mechanism f is a strategyproof, $\frac{23}{13}$- approximate mechanism.*

Proof sketch. The proof is similar in spirit to that of Theorem 1. We show that $\frac{sb(\mathbf{x},a)}{sb(\mathbf{x},f(\mathbf{x}))} \leq \frac{23}{13}$ for all possible facility locations $a \in I$ and profiles $\mathbf{x}$. Our proof works by analyzing the cases where $a \in [0, 1]$ and $a \in [1, 2]$. In each of these cases, we obtain two possible location profiles, such that at least one of them must lead to the worst case approximation ratio. We then show that the worst-case "a" in all these profiles is either 0, 1 or 2. This analysis results in studying 7 cases, and we show the bound of $\frac{23}{13}$ in each.

The approximation ratio of the mechanism above is tight: when there are two agents of different types, with the type 1 agent at 1 and the type 2 agent at 0, the optimal benefit is 3, whereas the mechanism's expected benefit is $\frac{39}{23}$, and the ratio is exactly $\frac{23}{13}$.

4 Characterization of Deterministic Mechanisms for the Obnoxious Facility Model

We now focus on the special case where $N_2 = \emptyset$, also called the obnoxious facility model. The assumption $N_2 = \emptyset$ will remain in effect for the rest of the paper. In this section, we characterize all deterministic strategyproof mechanisms. A similar result has been independently obtained in an unpublished paper (Han and Du 2012). We begin with a temporary, somewhat weak characterization of deterministic mechanisms, in terms of single agent deviations:

[3]In (Cheng, Yu, and Zhang 2013), the authors note that this mechanism is 2- approximate for the obnoxious facility model; this still holds true for the hybrid model.

Theorem 3 (Reflection Theorem). *For any deterministic mechanism f, agent $i \in N$, and partial location profile $\mathbf{x_{-i}}$, define $f_{\mathbf{x_{-i}}}(a) = f(a, \mathbf{x_{-i}})$ [4]. Then, the mechanism f is strategyproof iff each $f_{\mathbf{x_{-i}}}$ is of the following form: there exists (not necessarily distinct) $\alpha_{\mathbf{x_{-i}}}, \beta_{\mathbf{x_{-i}}} \in I$, such that $\beta_{\mathbf{x_{-i}}} \geq \alpha_{\mathbf{x_{-i}}}$ and:*

1. *$f_{\mathbf{x_{-i}}}(a) = \beta_{\mathbf{x_{-i}}}$ for $0 \leq a < \frac{\alpha_{\mathbf{x_{-i}}} + \beta_{\mathbf{x_{-i}}}}{2}$*
2. *$f_{\mathbf{x_{-i}}}(a) = \alpha_{\mathbf{x_{-i}}}$ for $\frac{\alpha_{\mathbf{x_{-i}}} + \beta_{\mathbf{x_{-i}}}}{2} < a \leq 2$*
3. *$f_{\mathbf{x_{-i}}}(\frac{\alpha_{\mathbf{x_{-i}}} + \beta_{\mathbf{x_{-i}}}}{2}) \in \{\alpha_{\mathbf{x_{-i}}}, \beta_{\mathbf{x_{-i}}}\}$*

If $\alpha_{\mathbf{x_{-i}}} \neq \beta_{\mathbf{x_{-i}}}$, we call $\frac{\alpha_{\mathbf{x_{-i}}} + \beta_{\mathbf{x_{-i}}}}{2}$ the reflection point of i for the partial profile $\mathbf{x_{-i}}$.

Proof sketch. The verification of strategyproofness is easy. On the other hand, let f be a strategyproof mechanism. Fix a location profile $\mathbf{x}$ and an agent i. Let $g = f_{\mathbf{x_{-i}}}$ and let $\beta = g(0)$, $S = \{a \in I : g(a) \neq \beta\}$ and $m = \inf S$. Furthermore, let $\alpha = 2m - \beta$. By definition, $g(a) = \beta$ for $a < m$. In the proof, we show that if $m \in S$, then $g(m) = \alpha$, and if $m \notin S$, then m is a limit point of $T := \{a \in I : g(a) = \alpha\}$ (and $g(m) = \beta$ by the definition of S). It remains to show that $g(a) = \alpha$ for all $a > m$. Assume otherwise for some $a' > m$. First, note that as $g(m)$ is either α or β, $g(a) \in [\alpha, \beta]$ for all $a \in I$ by strategyproofness. Note that within this range, α is the point furthest from a'. Thus, the agent has an incentive to deviate from a' to any point a'' for which $g(a'') = \alpha$; the existence of such a point is guaranteed, as either $g(m) = \alpha$ or $T \neq \emptyset$. Thus strategyproofness is violated.

As a corollary of the above theorem, we can deduce:

Corollary 1. *For any deterministic strategyproof mechanism f, and any $n \in \mathbb{N}$, $R_n^f = \{f_n(\mathbf{x}) : \mathbf{x} \in I^n\}$ is finite.*

Proof sketch. Let $\mathbf{x}$ be an arbitrary profile. For each agent, we use the reflection theorem to show that the agent is able to move to at least one of the endpoints without changing the facility location. Thus, we obtain a location profile $\mathbf{x}'$, which locates all the agents at the endpoints and $f(\mathbf{x}') = f(\mathbf{x})$. Therefore, for every $y \in R_n^f$, there exists a profile $\mathbf{z}$ s.t. $f(\mathbf{z}) = y$ and $\mathbf{z}$ locates all the agents at the endpoints; but there are only finitely many such profiles.

Now it is time for our strong characterization result. Consider the following definition:

Definition 2. *Let f be a deterministic mechanism s.t. $|R_n^f| \leq 2$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $R_n^f = \{\alpha_n, \beta_n\}$ s.t. $\beta_n \geq \alpha_n$ [5], and let $m_n = \frac{\alpha_n + \beta_n}{2}$. For any $n \in \mathbb{N}$, for every profile $\mathbf{x} \in I^n$, consider the partition of the agents $L_n^{\mathbf{x}} = \{i \in N : x_i < m_n\}$, $M_n^{\mathbf{x}} = \{i \in N : x_i = m_n\}$, and $E_n^{\mathbf{x}} = \{i \in N : x_i > m_n\}$. We say that f is a midpoint mechanism if it satisfies the following property: for any $n \in \mathbb{N}$, let $\mathbf{x}, \mathbf{y} \in I^n$ be any profiles s.t. $f(\mathbf{x}) = \beta_n$ and $f(\mathbf{y}) = \alpha_n$. If $\beta_n > \alpha_n$, then there exists an agent i which satisfies one of the following:*

[4]Note that when $i \neq j$, $\mathbf{x_{-i}}$ and $\mathbf{x_{-j}}$ are distinct objects, regardless of the values of their coordinates.

[5]$\alpha_n = \beta_n$ is possible.

(D-1) $i \in L_n^{\mathbf{x}}$ *and* $i \in M_n^{\mathbf{y}}$
(D-2) $i \in L_n^{\mathbf{x}}$ *and* $i \in E_n^{\mathbf{y}}$
(D-3) $i \in M_n^{\mathbf{x}}$ *and* $i \in E_n^{\mathbf{y}}$

This definition is simple to interpret: the mechanism can switch the facility location from right to left or from left to right only when an agent crosses the midpoint in the opposite direction.

In (Ibara and Nagamochi 2012), the authors show that for a strategyproof mechanism f, whenever R_n^f is a finite set, $|R_n^f| \leq 2$ [6]. Using that, we can now show:

Theorem 4. *A deterministic mechanism f is strategyproof iff it is a midpoint mechanism.*

Proof. Verifying strategyproofness is easy. Now, assume that f is a strategyproof mechanism. Fix $n \in \mathbb{N}$. By corollary 1, R_n^f is finite, and thus by Ibara's and Nagamochi's result, $|R_n^f| \leq 2$. If R_n^f is a singleton there is nothing to prove; thus, assume $|R_n^f| = 2$, and let $\alpha_n, \beta_n \in R_n^f$ s.t. $\beta_n > \alpha_n$. Let $\mathbf{x}, \mathbf{y} \in I^n$ s.t. $f(\mathbf{x}) = \beta_n$ and $f(\mathbf{y}) = \alpha_n$. Consider the sequence of profiles $\mathbf{z}^i$, defined for $i = 0, \ldots, n$ via $\mathbf{z}_j^i = \mathbf{x}_j$ if $j > i$ and $\mathbf{z}_j^i = \mathbf{y}_j$ otherwise. Assume no agent satisfies at least one of (D-1), (D-2) and (D-3). Then, when agent i deviates in $\mathbf{z}^{i-1}$ to create profile $\mathbf{z}^i$, he does not cross m_n from left to right (i.e. moving from $\mathbf{z}_i^{i-1} < m_n$ to $\mathbf{z}_i^i \geq m_n$ or $\mathbf{z}_i^{i-1} \leq m_n$ to $\mathbf{z}_i^i > m_n$). As the possible facility locations are α_n and β_n, m_n is his only candidate for reflection point in $\mathbf{z}_{-\mathbf{i}}^{i-1}$. Thus, the reflection theorem implies that he cannot change the facility location to α_n by deviating. Hence, $f(\mathbf{y}) = f(\mathbf{z}^n) = f(\mathbf{z}^0) = f(\mathbf{x})$, contradiction. $\square$

We note that Ibara and Nagmochi have characterized all anonymous mechanisms under the assumption that R_n^f is finite for all $n \in \mathbb{N}$, using what they called "valid threshold mechanisms". Our proofs easily translate to the anonymous case, and under anonymity, our midpoint mechanisms become equivalent to valid threshold mechanisms. Thus, our work allows the removal of the finite R_n^f assumption for the anonymous case as well.

5 Lower Bounds on Deterministic Mechanisms

We can use our characterization to obtain lower bounds on the possible approximation ratios for the maxisum and egalitarian objectives in the deterministic setting.

Theorem 5. *No deterministic strategyproof mechanism f can provide an approximation ratio better than 3 for the maxisum objective, even when $N_2 = \emptyset$.*

Proof sketch. Let f be a deterministic strategyproof mechanism, and let $n \in \mathbb{N}$ be even. The case of R_n^f being a singleton is easy, and by Theorem 4, if R_n^f is not a singleton, then $|R_n^f| = 2$. Consider the profile $\mathbf{x} \in I^n$ which locates agents 1 through $\frac{n}{2}$ at α_n, agents $\frac{n}{2} + 1$ through n at β_n, and assume without loss of generality that $f(\mathbf{x}) = \beta_n$. If we relocate agents 1 through $\frac{n}{2}$ to $m_n - \epsilon$, by Theorem 4 the facility is still located at β_n, but the resulting approximation ratio is $\frac{3(\beta_n - \alpha_n) - 2\epsilon}{\beta_n - \alpha_n + 2\epsilon}$. Sending $\epsilon \to 0$ gives us the required result.

By Theorem 1, our lower bound is best-possible. Our characterization can also be used to get a lower bound for the egalitarian objective:

Theorem 6. *No deterministic strategyproof mechanism f can provide a bounded approximation ratio for the egalitarian objective, even when $N_2 = \emptyset$.*

Proof. Assume $N_2 = \emptyset$. For any $n \geq 2$, $|R_n^f| \leq 2$ by theorem 4. Consider any profile which locates at least one agent at each point in R_n^f; any such profile leads to a social benefit of 0 for the mechanism, whereas the optimal benefit is positive. $\square$

6 Lower Bounds on Randomized Mechanisms

We begin with the maxisum objective. We provide a lower bound of $\frac{3}{2}$ on the best-possible approximation ratio obtainable by randomized strategyproof mechanisms.

Lemma 1. *For any randomized strategyproof mechanism f, there exists a randomized strategyproof mechanism f' which satisfies $P(f'(\mathbf{x}) \in \{0, 2, x_1, x_2, ..., x_n\}) = 1$ for each $\mathbf{x} \in I^n$ and has the same approximation ratio as f for the maxisum objective.*

Theorem 7. *No randomized strategyproof mechanism can provide an approximation ratio better than $\frac{3}{2}$ for the maxisum objective, even when $N_2 = \emptyset$.*

Proof sketch. Assume f is such a mechanism, and consider the case of 2 agents. By Lemma 1, we may assume that $P(f(\mathbf{x}) \in \{0, 2, x_1, x_2\}) = 1$ for all profiles $\mathbf{x} \in I^2$. As a convenient notation, let $p_q^{\mathbf{x}} = P(f(\mathbf{x}) = q)$. The proof proceeds as follows: in the profile $\mathbf{y}$ in which $y_1 = 1 - \epsilon$ and $y_2 = 2$, to beat the $\frac{3}{2}$ approximation ratio, we must have $p_0^{\mathbf{y}} > \frac{1}{2}$ for small enough $\epsilon > 0$. However, in that case, strategyproofness dictates that in the profile $\mathbf{z}$, in which $z_1 = 0$ and $z_2 = 0$, $p_0^{\mathbf{z}} > \frac{1}{2}$ as well (otherwise, $p_2^{\mathbf{z}} > \frac{1}{2}$, and we can use that to show that it is beneficial for agent 1 to deviate from $\mathbf{y}$ to $\mathbf{z}$). However, a symmetric argument gives that $p_2^{\mathbf{z}} > \frac{1}{2}$ as well, which leads to a contradiction.

We note that, when $N_2 = \emptyset$, Theorem 10 provides a family of randomized $\frac{3}{2}$- approximate strategyproof mechanisms for the maxisum objective. Hence, when $N_2 = \emptyset$, the above lower bound is best-possible.

Finally, we show the same lower bound for the egalitarian objective:

Theorem 8. *No randomized strategyproof mechanism can provide an approximation ratio better than $\frac{3}{2}$ for the egalitarian objective, even when $N_2 = \emptyset$.*

Proof sketch. Let $I = [0, M + 2]$ for some large M, and consider the profile $\mathbf{x}$ that locates one agent at 1 and $M + 1$, a large number of agents at 0 and $M + 2$, as well

[6] While they assume anonymity, the proof of this fact does not rely on that assumption.

as agents spread across $[1, M+1]$ which cover that interval so that there is at least one agent in every subinterval of length ϵ. For a mechanism f to have an approximation ratio $c < \frac{3}{2}$, we show that $p = P(f(\mathbf{x}) \in [1, M+1]) \leq \frac{\frac{1}{2}(1-\frac{1}{c})}{\frac{1}{2}-\frac{\epsilon}{2}}$. Assume without loss of generality that f locates the facility at $[0,1]$ with probability at least $\frac{1-p}{2}$. We now consider a profile $\mathbf{x}'$ obtained from $\mathbf{x}$ by moving agents from 0 to spread them along the interval $[0,1]$, so that now there is at least one agent in every subinterval of length ϵ of $[0, M+1]$. To maintain the approximation ratio c, we must have $P(f(\mathbf{x}') \in [0, M+1]) \leq \frac{\frac{1}{2}(1-\frac{1}{c})}{\frac{1}{2}-\frac{\epsilon}{2}}$. However, strategyproofness limits $P(f(\mathbf{x}') \in [M+1, M+2])$, as that interval is far from 0, and a high probability of locating the facility there will incentivize the agents located at 0 in $\mathbf{x}$ to misreport their locations as their locations in $\mathbf{x}'$ [7]. A careful analysis of these bounds leads to a contradiction.

7 Mutiple Locations Per Agent in the Obnoxious Model

In this section we follow the spirit of a suggestion in (Procaccia and Tennenholtz 2013) and study a generalized model, in which a single agent may be associated with more than one location. As this multiple location model is a generalization of our previous model, the lower bounds carry over; in particular, for the maxisum objective, we have lower bounds of 3 and $\frac{3}{2}$ on deterministic and randomized mechanisms respectively, even when $N_2 = \emptyset$. We show that when $N_2 = \emptyset$, we can find strategyproof mechanisms to match these lower bounds, despite the additional power given to the agents.

Our generalized model (for the definition of the model, we do not assume $N_2 = \emptyset$) can be obtained from our previous model via the following changes. First, let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$. A location profile is now $\mathbf{z} = (\mathbf{z^1}, \mathbf{z^2}, \ldots, \mathbf{z^n})$, where for each $i = 1, ..., n$, $\mathbf{z^i} = (z_1^i, z_2^i, \ldots, z_{k_i}^i) \in I^{k_i}$. A deterministic mechanism is a collection of functions $f = \{f_n^{\mathbf{k}} : n \in \mathbb{N}, \mathbf{k} \in \mathbb{N}^n\}$, such that $f_n^{\mathbf{k}} : I^{k_1} \times \ldots \times I^{k_n} \to I$ is a function that maps each location profile to a facility location. The benefit of agent i from facility location y is now defined as $B_i(\mathbf{z^i}, y) = \sum_{j=1}^{k_i} B_i(z_j^i, y)$, where $B_i(x, y)$ is $|x-y|$ for a type 1 agent and $2 - |x-y|$ for a type 2 agent. The maxisum objective is $\sum_{i=1}^{n} B_i(\mathbf{z^i}, y)$ as usual. The rest of the notation carries over, and the adjustment to the randomized model is easy and left to the reader. For the approximation ratio, we note that the possible instances of the problem include all possible options for both n and $\mathbf{k}$.

First, we provide a 3- approximate strategyproof deterministic mechanism for the case where $N_2 = \emptyset$.

Theorem 9. *Let* $R^* = \{i : \frac{\sum_{j=1}^{k_i} z_j}{k_i} \leq 1\}$, $L^* = \{i : \frac{\sum_{j=1}^{k_i} z_j}{k_i} > 1\}$. *Let f be the mechanism which locates the facility at 2 if $\sum_{i \in R^*} k_i \geq \sum_{i \in L^*} k_i$ and at 0 otherwise. This mechanism is strategyproof and 3- approximate for the maxisum objective when $N_2 = \emptyset$.*

[7] This argument can be converted into a similar argument regarding single agent deviations.

Note that when $k_i = 1$ for all i, this mechanism reduces to the mechanism proposed in (Cheng, Yu, and Zhang 2013).

Finally, we define a class of randomized strategyproof mechanisms that provide a $\frac{3}{2}$- approximation ratio when $N_2 = \emptyset$ and show that it is nonempty.

Theorem 10. *Let f be a randomized mechanism that, for a profile $\mathbf{z}$, locates the facility at 0 with probability $p_{\mathbf{z}}$ and at 2 with probability $(1 - p_{\mathbf{z}})$. Then, when $N_2 = \emptyset$, the following conditions on $p_{\mathbf{z}}$ are sufficient to make the mechanism strategyproof and $\frac{3}{2}$- approximate:*

1. *$p_{\mathbf{z}}$ is increasing in $\sum_{i \in L^*} k_i$ and decreasing in $\sum_{i \in R^*} k_i$.*
2. $\frac{1}{3} + \frac{1}{6} \cdot \frac{\sum_{i \in L^*} k_i}{\sum_{i \in R^*} k_i} \geq p_{\mathbf{z}} \geq \frac{2}{3} - \frac{1}{6} \cdot \frac{\sum_{i \in R^*} k_i}{\sum_{i \in L^*} k_i}$ *(if $\sum_{i \in R^*} k_i = 0$, the leftmost term is ∞; if $\sum_{i \in L^*} k_i = 0$, the rightmost term is $-\infty$).*

Furthermore, the class of mechanisms of this form is nonempty.

Proof sketch. The fact that $p_{\mathbf{z}}$ is increasing in $\sum_{i \in L^*} k_i$ and decreasing in $\sum_{i \in R^*} k_i$ clearly implies strategyproofness. The bounds on p_z follow from a careful analysis of the profile which locates $z_j^i \approx 1$ for all $i \in L^*$ and $z_j^i = 0$ for all $i \in R^*$, and the profile which locates $z_j^i = 1$ for all $i \in R^*$ and $z_j^i = 2$ for all $i \in L^*$ (for all j), which are the profiles that lead to the worst case approximation ratios. The nonemptiness of the class comes from showing that $p_{\mathbf{z}} = \max\{\frac{2}{3} - \frac{1}{6} \cdot \frac{\sum_{i \in R^*} k_i}{\sum_{i \in L^*} k_i}, 0\}$ satisfies the above properties.

It is worth noting that the randomized mechanism given in (Cheng, Yu, and Zhang 2013), for the special case of $k_i = 1$ for all i, falls into the category of mechanisms we defined here.

8 Future Research

The immediate question stemming from our results is what further improvement can be achieved in the approximation ratio for the hybrid model by using randomization. Another question is what can be done in the randomized setting for the egalitarian objective. Other directions include characterization and bounds for strategyproof mechanisms on topologies different than the interval, and for objectives other than maxisum and egalitarian. We are exploring these questions in an ongoing work.

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