The Thirty-Second AAAI Conference
on Artificial Intelligence (AAAI-18)

Expected Utility with Relative Loss Reduction: A Unifying Decision Model for Resolving Four Well-Known Paradoxes

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Abstract

Some well-known paradoxes in decision making (*e.g.*, the Allais paradox, the St. Petersburg paradox, the Ellsberg paradox, and the Machina paradox) reveal that choices conventional expected utility theory predicts could be inconsistent with empirical observations. So, solutions to these paradoxes can help us better understand humans decision making accurately. This is also highly related to the prediction power of a decision-making model in real-world applications. Thus, various models have been proposed to address these paradoxes. However, most of them can only solve parts of the paradoxes, and for doing so some of them have to rely on the parameter tuning without proper justifications for such bounds of parameters. To this end, this paper proposes a new descriptive decision-making model, *expected utility with relative loss reduction*, which can exhibit the same qualitative behaviours as those observed in experiments of these paradoxes without any additional parameter setting. In particular, we introduce the concept of relative loss reduction to reflect people's tendency to prefer ensuring a sufficient minimum loss to just a maximum expected utility in decision-making under risk or ambiguity.

Introduction

Decision-making is a process of selecting one among available choices (Russell et al. 2010). In real-world applications, decisions often have to be made under risk or ambiguity (Ma, Xiong, and Luo 2013; Luo, Zhong, and Leung 2015). Here, decision making under risk means that the consequence of a decision is uncertain but the probability of each possibility is known, while decision making under ambiguity means even the probability of each possibility is unclear. Since such uncertainties are inevitable in real-world applications, this topic is a central concern in decision science (Tversky and Kahneman 1992; Gul and Pesendorfer 2014) and artificial intelligence (Dubois, Fargier, and Perny 2003; Luo and Jennings 2007; Ma, Luo, and Jiang 2017). Perhaps the most well-known theory for decision-making under uncertainty is expected utility theory (Neumann and Morgenstern 1944). Specifically, it is about how to make optimal decisions under risk. It gives a normative interpretation for researchers in economics and artificial intelligence to model applications related to rational agents.

Unfortunately, recent empirical studies, such as the Allais paradox (Allais and Hagen 2013), the St. Petersburg paradox (Joyce 2011), the Ellsberg paradox (Ellsberg 1961), and the Machina paradox (Machina 2009), have observed a number of violations (*i.e.*, humans' choices in reality deviated from those predicted by classic expected utility theory (Neumann and Morgenstern 1944)). At the same time, in many artificial intelligence applications such as recommendation system, supply chain management, policy making, human-computer negotiation, and security surveillance system, human decision-making behaviour indeed is an essential concern and computers often need to predict precisely human decisions under uncertainty. For example, if a government policy-making support system cannot understand how the people will respond to a policy, it is questionable to recommend the policy to the government (Zhan et al. 2018).

Since the classic expected utility theory cannot predict human decisions observed in these paradoxes, various generalised expected utility theories and ambiguity decision models have been proposed to account the paradoxes. However, most of them can only resolve parts of the paradoxes, and some of them (*e.g.*, (Tversky and Kahneman 1992; Quiggin 2012; Gul and Pesendorfer 2014)) can only solve the paradoxes by adding some extra parameters, which means that these models only work in cases that the bounds of parameter variations are already known. Unfortunately, this makes it hard to evaluate the predication power of the models in a new decision-making problem (more detailed discussion can be found in related work section).

To address this issue, this paper introduces a new concept, called *relative loss reduction*, which is based on the human nature of loss aversion, which has been convincingly confirmed by many psychological and economic experiments (Kahneman 2003). In psychology, loss aversion refers to people's tendency to prefer avoiding losses to acquiring equivalent gains (Kahneman 2003). In this paper, we interpret the meaning of loss in two cases: (i) the difference between the minimum utilities of any two choices; and (ii) the span of the expected utility interval of a choice. In the first case, the loss aversion means a decision maker prefers a choice with higher minimum utility, *ceteris paribus*; and

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in the second case, the loss aversion means a decision maker prefers a choice with less ambiguous, *ceteris paribus*. And since the two interpretations are based on the comparison of multiple values and normally a decision maker cares about the reduction of loss in a decision making problem, we call these two interpretations as *relative loss reduction*.

Then, we construct a formal descriptive decision-making model to explore how this concept helps to make an optimal choice in decision-making and address three well-documented deviations from expected utility theory without any extra parameter:

(i) Allais-style evidence: a decision maker prefers the choice with certain reward to the choice under risk, but reverses this ordering if both choices are mixed with an undesirable possible consequence. Thus, it shows violations of the independence axiom.

(ii) St. Petersburg-style evidence: a decision maker only wants to pay a very small amount of money to play a particular lottery game that has a random consequence with a huge expected utility.

(iii) (iii) Ellsberg-style evidence: a decision maker has the nature of ambiguity aversion in case that the precise probabilities of some possible consequences of the choices are unavailable.

The remainder of this paper is organised as follows. Firstly, we give the formal definition of decision problems under uncertainty, and recap the methods to obtain expected utilities and expected utility intervals. Secondly, we present our decision-making model that reflects the human nature of loss aversion in making an optimal choice and reveal some insights into our model. Thirdly, we further validates our model by showing that it can resolve four well-known paradoxes. Fourthly, we discuss the related work. Finally, we conclude the paper with further work.

Problem Definition

In this section, we give a formal definition for the problem of decision-making under uncertainty (including risk and ambiguity), and recap the notions of expected utility (Neumann and Morgenstern 1944), and expected utility interval (Ghirardato, Maccheroni, and Marinacci 2004).

Definition 1 *A decision problem under uncertainty is a 4-tuple (S, C, u, Q),*[1] *where:*

(i) S is a set of all the possible states;

(ii) C is a convex set of all the available choices. That is, if $c_1, c_2 \in C$ and $\lambda \in [0, 1]$, then a uncertain choice $\lambda c_1 + (1-\lambda)c_2$, meaning to choose c_1 with a chance of λ and choose c_2 with a chance of $1-\lambda$, is also in C.

(iii) u is a mapping from $C \times S$ to $\mathbb{R}$, representing the utility of making choice $c \in C$ in state $s \in S$.

(iv) $\Delta(S) = \{p \mid p : S \to [0, 1], \sum_{s_i \in S} p(s_i) = 1\}$, i.e., *the set of all the probability distributions p over S.*

(v) Q is a subset of $\Delta(S)$, representing the probability assignments for each state $s \in S$.

[1]This paper considers the decision problem with choice-state independence to simplify the issue. However, in future it is worth studying whether or not that our model can be applied to the cases of choice-state dependence as well.

Then by Definition 1, a decision problem under risk is a 4-tuple $(S, C, u, \{p\})$, where $p \in Q$ is a unique probability distribution for the decision maker to represent the precise probability value for each state. Whilst, a decision problem under ambiguity is a 4-tuple (S, C, u, Q), where $|Q| > 1$, representing multiple probability values for some states.

The most widely used model of decision making under uncertainty is expected utility theory (Neumann and Morgenstern 1944), in which expected utility of a choice is given as follows:

$$EU(c) = \sum_{s_i \in S} p(s_i)u(c, s_i). \tag{1}$$

For a decision problem under ambiguity, Ghirardato, Maccheroni, and Marinacci (2004) introduce the concept of expected utility interval of a choice c as follows:

Definition 2 *Given a decision problem under ambiguity (S, C, u, Q), the expected utility interval of a choice c, denoted as $EUI(c) = [\underline{E}(c), \overline{E}(c)]$, is given by:*

$$\underline{E}(c) = \min_{p \in Q} EU_p(c), \tag{2}$$

$$\overline{E}(c) = \max_{p \in Q} EU_p(c). \tag{3}$$

Decision-Making with Relative Loss Reduction

This section will reveal some insights into the concept of relative loss reduction in decision-making under risk and decision making under ambiguity.

As mentioned previously, the idea of relative loss reduction is based on the human nature of loss aversion. Usually, loss aversion means that the focus is on the real loss in a decision making problem. For example, one who losed $\$100$ loses more satisfaction than another who gains satisfaction from a windfall of $\$100$. Nevertheless, this human nature occurs not only in the case that the real consequence of a choice is negative, but also in the case the possible consequence of a choice may be worse than the decision maker's expectation. For example, in decision making under risk, suppose choice a is to win $\$80$ with a chance of 50% and to win $\$20$ with a chance of 50%; and choice b is to win $\$100$ with a chance of 50% and get nothing with a chance of 50%. Then a decision maker, who makes choice b, suffers an anticipated loss aversion: if choice b turns out to be zero, I will lost at least $\$20$ since it is the minimum utility I can obtain by making choice a. Similarly, for decision making under ambiguity, if choice a is to win $\$100$ with a chance between 0 and $2/3$ and to get nothing otherwise, and choice b is to win $\$100$ with a chance of $1/3$ and to get nothing otherwise, then a decision maker, who makes choice a, suffers another anticipated loss aversion: if the real probability of wining $\$100$ is 0, I will loss $\$100$ with a chance of $1/3$ since it is the minimum probability of obtaining $\$100$ by making choice b.

According to the above intuitions, we can interpret two types of loss aversion as follows. (i) *Worst Case Loss Aversion:* This happens when the worst consequence of the choice made is worse than the worst consequence of other choices, and it is due to the uncertainty of the states of the world and the comparison of multiple choices. (ii) *Ambiguity Loss Aversion:* This happens when the expected utility of a choice is unclear (*i.e.,* it is interval-valued), and it is due to the uncertainty of the probability of some states of world and the comparison of multiple choices. Since the two types of loss aversion are based on the comparison of multiple choices and the loss aversions lead the decision maker to make a choice with less loss, we call the concept *relative loss reduction*.

Based on the above understanding of relative loss reduction, in the following we will establish a formal model to predict the behaviour of an anticipated loss aversion proneness decision maker.

Risky Expected Utility with Relative Loss Reduction

First, we discuss a decision problem under risk. For this problem, since the expected utility of a choice is determined by expected utility theory, we only need to consider the influence of the worst case loss aversion. Formally, we can define expected utility with relative loss reduction (RLR) for decision problems under risk (or simple *risky expected utility with RLR*) and accordingly set the preference ordering as follows:

Definition 3 *For a decision problem under risk $(S, C, u, \{p\})$, let $s_i \in S$ $(j = 1, \ldots, n)$ be a state of the world, $u(c, s_i)$ be the utility of a choice $c \in C$ at state $s_i \in S$, and $EU(c)$ be the expected utility value given by formula (1). Then the risky expected utility with relative loss reduction (RLR) of a choice $c \in C$, denoted as $L_r(c)$, is given by:*

$$L_r(c) = EU(c) - (\max_{c' \in C} \min_{s_i \in S} u(c', s_i) - \min_{s_j \in S} u(c, s_j)). \quad (4)$$

For any two choices c_1 and c_2, the preference ordering based on risky expected utility with RLR, denoted as $\succeq_r$, is defined as:

$$c_1 \succeq_r c_2 \Leftrightarrow L_r(c_1) \geq L_r(c_2). \quad (5)$$

In Definition 3, the loss aversion in risk is represented as

$$\max_{c_i \in C} \min_{s_j \in S} u(c_i, s_j) - \min_{s_k \in S} u(c_1, s_k),$$

which actually is the distance between the maximum minimum utility of any choice $c_i \in C$ and the minimum utility of a selected choice c_1. Moreover, since the effect only occurs when the minimum consequence of a choice is worse than other choices, we can find that for a choice with the maximum minimum utility, the worst case loss aversion should be 0. And $L_r(c_1)$ shows the reduced expected utility of a choice under the influence of anticipated worst case loss aversion. Finally, since for a decision making problem (S, C, u, Q) with $|Q| > 1$ the expected utility is interval-valued, more factors should be considered when we extend Definition 3 to the case of ambiguity (this will be discussed in next section).

Let us consider the following example:

Example 1 *Alice is thinking about investing $\$240K$ in stock. And she considers a stock offered by Bob (an investment advisor) that give she $\$150K$ with a chance of 30% and $\$300K$ with a chance of 70% after one year. Clearly, by formula (1), the expected utility is $255K$. However, by Definition 3, since the minimum expected utility of rejecting this offer is $240K$ and that of accepting the offer is $\$150K$, after considering the loss aversion in risk (*i.e.*, $240K - 150K = 90K$), the risky expected utility with RLR of accepting the offer is $165K$ (*i.e.*, $255K - 90K$), while that of rejecting the offer is $240K$. Thus, Alice should reject the offer.*

Now, in the following theorem we reveal some properties about the preference ordering set in Definition 3:

Theorem 1 *Let C be a finite choice set, S be a state set, and $EU(c)$ be the expected utility of choice c. Then for any $c_1, c_2, c_3 \in C$, the preference ordering $\succeq_r$ satisfies:*

1. ***Weak Order:*** *(i) Either $c_1 \succeq_r c_2$ or $c_2 \succeq_r c_1$; and (ii) if $c_1 \succeq_r c_2$ and $c_2 \succeq_r c_3$, then $c_1 \succeq_r c_3$.*
2. ***Archimedean Axiom:*** *If $c_1 \succeq_r c_2$ and $c_2 \succeq_r c_3$, then there exist $\lambda, \mu \in (0, 1)$ such that*
$$\lambda c_1 + (1 - \lambda)c_3 \succeq_r c_2 \succeq_r \mu c_1 + (1 - \mu)c_3.$$
3. ***Monotonicity:*** *If for any $q \in \Delta(S)$ satisfying $q(s_i) = 1$ $(s_i \in S)$, we have $c_1 \succeq_r c_2$, then $c_1 \succeq_r c_2$ for any $p \in \Delta(S)$.*
4. ***Sen's property:*** *If $c_1 \succeq_r c_2$ in a choice set C, then for any choice set C', such as $c_1, c_2 \in C'$, we have $c_1 \succeq_r c_2$.*
5. ***Limited Constant Independence:*** *If $c_1 \succeq_r c_2$ and c^* is a choice with certain consequence that satisfies $EU(c^*) \geq \min_{i=1,2} \min_{s_j \in S} \{u(c_i, s_j)\}$, then*
$$\lambda c_1 + (1 - \lambda)c^* \succeq_r \lambda c_2 + (1 - \lambda)c^*$$
for any $\lambda \in (0, 1]$.
6. ***Certainty Effect:*** *If $EU(c_1) = EU(c^*)$ and c^* is a choice with certain consequence, then $c^* \succeq_r c_1$.*

Proof: We check the properties in this theorem one by one.

(i) Since $L_r(c_1) \geq L_r(c_2)$ or $L_r(c_1) \leq L_r(c_2)$ holds for any $c_1, c_2 \in C$, by Definition 3, we have $c_1 \succeq_r c_2$ or $c_2 \succeq_r c_1$. Moreover, suppose $c_1 \succeq_r c_2$ and $c_2 \succeq_r c_3$. Then by Definition 3, we have $L_r(c_1) \geq L_r(c_2)$ and $L_r(c_2) \geq L_r(c_3)$. As a result, $L_r(c_1) \geq L_r(c_3)$. Thus by Definition 3, we have $c_1 \succeq_r c_3$. So, property 1 holds.

(ii) Suppose $c_1 \succeq_r c_2$ and $c_2 \succeq_r c_3$. Then by Definition 3, we have $L_r(c_1) \geq L_r(c_2)$ and $L_r(c_2) \geq L_r(c_3)$. Since $L_r(c_i) \in \Re$ for $i \in \{1, 2, 3\}$, by the continuity of the real number, there exists $\lambda, \mu \in (0, 1)$, such that

$$\lambda L_r(c_1) + (1-\lambda) L_r(c_3) \geq L_r(c_2) \geq \mu L_r(c_1) + (1-\mu) L_r(c_3).$$

Thus, by Definition 3 and the choice set is convex, property 2 holds.

(iii) Suppose for any $q \in \Delta(S)$ satisfying $q(s_i) = 1$ $(s_i \in S)$, we have $c_1 \succeq_r c_2$. Then by Definitions 1 and 3, we have $u(c_1, s) \geq u(c_2, s)$ for $\forall s \in S$. Thus, by formula (1), we have $EU(c_1) \geq EU(c_2)$. Hence, by Definitions 3 and the fact that

$\min_{s_i \in S} u(c_1, s_i) \geq \min_{s_j \in S} u(c_2, s_j)$, we can obtain $c_1 \succeq_r c_2$. Thus, property 3 holds.

(iv) Let $t = \max_{c_i \in C} \min_{s \in S} u(c_i, s)$ and $t' = \max_{c_j \in C'} \min_{s \in S} u(c_j, s)$. Then by Definition 3 and $c_1 \succeq_r c_2$ in a choice set C, we have

$$EU(c_1) - (t - \min_{s \in S} u(c_1, s)) \geq EU(c_2) - (t - \min_{s \in S} u(c_2, s)).$$

Hence, from the fact that $t = t' - (t' - t)$, we know that $L_r(c_1) \geq L_r(c_2)$ in choice set C'. Thus, property 4 holds.

(v) Let $\min_{s \in S} u(c_1, s) = k_1$, $\min_{s \in S} u(c_2, s) = k_2$, $EU(c^*) = a$, and $\max_{c_i \in C} \min_{s \in S} u(c_i, s) = t$, where $t > \lambda k_1 + (1-\lambda)a$ and $t > \lambda k_2 + (1-\lambda)a$. Then, by $c_1 \succeq_r c_2$ and Definition 3, we have:

$$\begin{aligned} & \lambda EU(c_1) + (1-\lambda)a - (t - (\lambda k_1 + (1-\lambda)a)) \\ \geq \;\; & \lambda EU(c_2) + (1-\lambda)a - (t - (\lambda k_2 + (1-\lambda)a)). \end{aligned}$$

Thus, by formula (5), property 5 holds.

(vi) Let $EU(c_1) = EU(c^*) = a$ and $\min_{s \in S} u(c_1, s) = k_1$. Then, by formula (1), $a \geq k_1$. Hence, by Definition 3, we have

$$a - (a - a) \geq a - (a - k_1).$$

Thus, by formula (5), we have $c^* \succeq_r c_1$. So property 6 holds. □

Properties 1-3 in the above theorem are the standard axioms in expected utility theory (Neumann and Morgenstern 1944). The *first* means a preference ordering can compare any pair of choices and satisfies transitivity. The *second* works like a continuity axiom on preferences, asserting that no choice is either infinitely better or infinitely worse than any other choices. The *third* is the requirement of monotonicity, asserting that if a decision maker does not think the utility of one choice is worse than the potential obtained utility of another choice on each state of world, then the former choice is conditionally preferred to the latter. And property 4 is implied by expected utility theory, meaning eliminating some of the unchosen choice should not affect the optimal choice.

The rest of the properties in the above theorem are our own. The *fifth* requires that the ranking of choices is not affected by mixing a choice with certain consequence, which has a higher utility than the minimum utilities of the choices. Intuitively the violations of independence should be due only to the worst case utility, rather than the uncertainty of states. So, the property gives the essence of worst case loss aversion: decision makers really care about the minimum utility they could obtain by making a choice. The *sixth* means the decision maker prefers a choice with a sure gain to risky one if they have the same expected utility. In fact, the property somehow describes the *certainty effect* (Kahneman 2011) for decision makers in real-world applications.

Ambiguous Expected Utility with Relative Loss Reduction

Now, we turn to decision problems under ambiguity. In this case, a decision maker manifests a worst case loss aversion as well as an ambiguity loss aversion.

For the first type of loss aversion in decision making under ambiguity, based on the maximum expected utility (see Definition 2) and Definition 3, its effect can be measured by:

$$\overline{E}(c) - (\max_{c' \in C} \min_{s_i \in S} u(c', s_i) - \min_{s_j \in S} u(c, s_j)), \quad (6)$$

where $\overline{E}(c)$ is the maximum expected utility of choice c.

Since the excepted utility interval is determined by the uncertainty of the probability as well as the utility of the consequences of a choice, the second type of loss aversion is determined by the ambiguity degree of the expected utility as well as the difference between maximum expected utility and minimum expected utility. For the ambiguity degree, it means the extent of the uncertainty about the probability distribution with respect to a choice.

Let us consider the following example:

Example 2 (Example 1 continued) *After Alice rejects the first offer, Bob offers her two new stocks whose prices heavily depend on the monetary policy. According to the market survey of the trend of monetary policy, there is one in three chances that the government will adopt a normal monetary policy (n) and there is two in three chances that the government will change the monetary policy, it is either being expansionary (e) or contractionary (c). Then, (i) c_1 is to get $\$300K$ if the monetary policy is normal, and get $\$240K$ otherwise; and (ii) c_2 is to win $\$300K$ if the monetary policy is expansionary, and get $\$240K$ otherwise.*

Clearly, in this example, state set S is $\{n, e, c\}$. Since it is unclear how many chances the monetary policy will be expansionary or contractionary, there exist multiple values of the probability of a state (*e.g.*, the probability of expansionary monetary policy can be any value in $[0, 2/3]$). Thus, the probability distribution over the possible states is uncertain. In order to consider its ambiguity degree, we first need to define the structure of the multiple probability distributions over the state set in such a decision-making under ambiguity as follows:

Definition 4 *For a decision problem under uncertainty (S, C, u, Q), a partition $[S] = \{S_1, \ldots, S_n\}$ of the state set S is a structure partition of the decision problem that reveals the essence of the uncertain probability distribution over state set S if it satisfies:*[2]

- *there exists a precise probability function $q : [S] \to [0, 1]$, such that $\sum_{S_i \in [S]} q(S_i) = 1$;*
- *for any $S_i \in [S]$, $S_i = \{s_k, \ldots s_l\}$ and for any probability distribution $p \in Q$ over all states in S_i, we have $\sum_{s \in S_i} p(s) = q(S_i)$; and*
- *for any $|S_i| > 1$ ($S_i \in [S]$), each nonempty proper subset of S_i has multiple probability values.*

In fact, inspired by the definition of mass function of D-S theory, Definition 4 defines a unique probability distribution q over $[S]$ for Q. By Definition 4, in Example 2, suppose $\{n, e, c\}$ is the state set, then the structure partition is

[2]A partition $[S] = \{S_1, \ldots, S_n\}$ means that (i) $S_i \neq \emptyset$ ($i = 1, \ldots, n$), (ii) $S_1 \cup \cdots \cup S_n = S$, and (iii) $S_i \cap S_j = \emptyset$ for any $S_i \neq S_j$ and $S_i, S_j \in \{S_1, \ldots, S_n\}$.

$[S] = \{\{n\}, \{e, c\}\}$ with the probability 1/3 for $\{n\}$ and 2/3 for $\{e, c\}$. However, the partition of $\{\{n\}, \{e\}, \{c\}\}$ is not a structure partition, since there does not exist a precise probability value for $\{e\}$ or $\{c\}$. And $\{\{n, e, c\}\}$ is not a structure partition either, because the probability of subset $\{n\}$ or $\{e, c\}$ has a unique value.

Clearly, by $\sum_{s_i \in S} p(s_i) = 1$ and Definition 4, we can easily prove that for any decision problem under uncertainty, there exists a structure partition. Now, based on Definition 4, we can formally define the ambiguity degree of a given choice as follows:

Definition 5 *For a decision problem under uncertainty* (S, C, u, Q)*, let* $[S] = \{S_1, \ldots, S_n\}$ *be a structure partition,* $q(S_i)$ $(i = 1, \ldots, n)$ *be the precise probability value assigned to* $S_i \in [S]$*,* $U_c = \{u(c, s) \mid s \in S\}$ *be the set of all the possible utilities of making choice c, and* $V_i = \{u(c, s_j) \mid s_j \in S_i, S_i \in [S]\}$ *be the set of potentially obtained utilities of state set* $S_i \in [S]$*. Then the ambiguity degree of choice c, denoted as* δ_c*, is given by:*

$$\delta_c = \sum_{i=1}^{n} q(S_i) \frac{\log_2 |V_i|}{\log_2 |U_c|}. \tag{7}$$

In fact, Definition 5 is inspired by the generalised Hartley measure for non-specificity (Dubois and Prade 1985) in D-S theory. By $q(S_i) \log_2 |V_i|$, it means that the higher the probability value and the more number of the potentially obtained utilities involved in the probability value, the more ambiguity the decision maker suffers in a state set S_i. Finally, for Example 2, by Definition 5, we can find $\delta_{c_1} = 0$ and $\delta_{c_2} = 2/3$. Clearly, the greater the ambiguity degree of a choice, the higher the value that is to be assigned to the minimum expected utility to show the ambiguity loss aversion the decision maker manifests.

Together with the worst case loss aversion in decision making under ambiguity (which is defined in formula (6)), we can formally define an expected utility with relative loss reduction in decision making under ambiguity (or simple ambiguous expected utility with RLR) and accordingly set the preference ordering as follows:

Definition 6 *For a decision problem under ambiguity* (S, C, u, Q) *with* $|Q| > 1$*. The ambiguous expected utility with RLR of choice c, denoted as* $L_a(c)$*, is given by:*

$$L_a(c) = \overline{E}(c) - \delta_c(\overline{E}(c)–\underline{E}(c)) - (\max_{c' \in C} \min_{s_i \in S} u(c', s_i) - \min_{s_j \in S} u(c, s_j)). \tag{8}$$

Then, for any two choices c_1 *and* c_2*, the preference ordering based on the ambiguous expected utility with RLR, denoted as* $\succeq_a$*, is defined as follows:*

$$c_1 \succeq_a c_2 \Leftrightarrow L_a(c_1) \geq L_a(c_2). \tag{9}$$

By Definition 6, we can define two types of loss aversion effect in a decision problem under ambiguity with one formula. That is, $\overline{E}(c)$ in formula (8) says that a decision maker expects the maximum expected utility in a given decision problem. However, since the maximum expected utility is obtained in case of ambiguity, the decision maker also reduces his evaluation of the choice according to his nature of the ambiguity loss aversion. And $\delta_c(\overline{E}(c)–\underline{E}(c))$ means that the ambiguity loss aversion is determined by the ambiguity degree and the distance of maximum expected utility and minimum expected utility. Finally, $\max_{c' \in C} \min_{s_i \in S} u(c', s_i) - \min_{s_j \in S} u(c, s_j)$ represents the worst case loss aversion of the decision maker.

Let us consider the following example:

Example 3 (Example 2 continued) *Since these two new stocks offered by Bob suffer no loss of capital, Alice only needs to decide which one she should accept. First, she calculates the expected utility intervals of both choices by Definition 2, and obtain* $EUI(c_1) = 260$ *and* $EUI(c_2) = [240, 280]$*. Then, after considering worst case loss aversion and ambiguity loss aversion of herself, she decides to make her decision based on the ambiguous expected utility with RLR. Thus, by Definition 5, she obtains* $\delta_{c_1} = 0$ *and* $\delta_{c_2} = 2/3$*. So, by the fact that the maximin utility of these two choices is* $240K$*. As a result, by Definition 6, she has* $L_a(c_1) = 260$ *and* $L_a(c_2) = 253.3$*. Thus, she chooses* c_1*.*

Now, we can show the relationship between the risky expected utility with RLR and the ambiguous expected utility with RLR by the following theorem:

Theorem 2 *Risky expected utility with RLR is a special case of ambiguous expected utility with RLR.*

Proof: For a decision problem under risk $(S, C, u, \{p\})$, suppose $p(s_i)$ is a precise probability value assigned to a state $s_i \in S$, $u(c, s_i)$ be the utility of the consequence that makes choice c in state s_i. Then by Definition 2, we have:

$$\underline{E}(c) = \overline{E}(c) = EU(c) = \sum_{s_i \in S} p(s_i) u(c, s_i).$$

Moreover, by Definitions 4 and 5, we have $\delta_c = 0$. Therefore, for a given choice c, by Definition 6, we have

$$L_a(c) = EU(c) - \sum_{s_i \in S} p(s_i)[EU(c) - u(c, s_i)] = L_r(c).$$

□

By Theorem 2, it means that we can use Definition 6 to obtain the expected utility with RLR of each choice for any decision problem under uncertainty. Thus, without losing generality, we can use $L_a(c)$ to represent the expected utility with relative loss reduction under uncertainty.

Also, the following theorem gives some properties for ambiguous expected utility with RLR:

Theorem 3 *Let C be a finite choice set and the interval-valued expected utility of choice* $c_i \in C$ *be* $EUI(c_i) = [\underline{E}(c_i), \overline{E}(c_i)]$*, and its ambiguity degree be* δ_{c_i}*. Then for two choices* c_1 *and* c_2 *with the same minimum utility, the binary relation* $\succeq_a$ *over C satisfies:*

(i) if $\underline{E}(c_1) > \underline{E}(c_2)$*,* $\overline{E}(c_1) > \overline{E}(c_2)$*, and* $\delta_{c_1} \leq \delta_{c_2}$*, then* $c_1 \succ_a c_2$*;*

(ii) if $\underline{E}(c_1) \geq \underline{E}(c_2)$ *and* $\delta_{c_1} = \delta_{c_2} = 1$*, then* $c_1 \succeq_a c_2$*; and*

(iii) if $EUI(c_1) = EUI(c_2)$ *and* $\delta_{c_1} < \delta_{c_2}$*, then* $c_1 \succ_a c_2$*.*

Table 1: Allais' Decision Situation Design

g_1		g_2		g_3		g_4	
Gain	p	Gain	p	Gain	p	Gain	p
\$100	100%	\$100	89%	0	89%	0	90%
		0	1%	\$100	11%		
		\$500	10%			\$500	10%

Proof: By Definition 6 and the fact that two choices c_1 and c_2 have the same minimum utility, we can obtain the result of Theorem 3 directly. $\square$

In fact, for two choices with the same minimum utility, property (i) of Theorem 3 means that if the ambiguity degree of a choice is not more than that of another, and the worst and the best expected utility of this choice is better than those of another respectively, this choice should be made. And property (ii) of Theorem 3 means that in the case of absolute ambiguity, a decision maker should take maximin attitude (*i.e.*, compare their minimum expected utilities and choose the best one). Finally, property (iii) reveals the relation between ambiguity degree and the preference ordering. That is, a decision maker should make a choice with less ambiguous, *ceteris paribus*.

Paradox Analysis

This section resolves four paradoxes using our model.

The Allais paradox (Allais and Hagen 2013) is a choice problem that shows expected utility theory is problematic. Suppose a decision maker needs to make a choice between two gambles: g_1 and g_2, or g_3 and g_4. The payoff for each gamble in each experiment is as shown in Table 1. Allais discovered that most participants picked g_1 rather than g_2, or alternatively most participants picked g_4 rather than g_3. However, this result is inconsistent with what expected utility theory predicts.

Using our model, by Definition 3, we have:

$$L_r(g_1)=100,\ L_r(g_2)=139-(100-0)=39,$$
$$L_r(g_3)=11-(0-0)=11,\ L_r(g_4)=50-(0-0)=50.$$

Thus, for a decision maker, we have $g_1 \succ_r g_2$ and $g_4 \succ_r g_3$. This result is the same as the observation of Allais. In fact, our model is able to not only resolve the Allais paradox, but also reveal the limitation of the influence about certainty effect. For instance, if g_2 has a chance of 89% to win \$100, a chance of 1% to win \$0, and a chance of 10% to win \$50,000, there is no doubt that most of people prefer g_2 to g_1. And if g_2 has a chance of 89% to win \$100, a chance of 1% to win \$70, and a chance of 10% to win \$500, people should also prefer g_2 to g_1.

The St. Petersburg game (Joyce 2011) is played by flipping a fair coin until it comes up tails, and the total number of flips n determines the prize, which equals $\$2^n$. Then, by formula (1), the expected utility of this game is $\sum_{n=1}^{\infty} \frac{1}{2^n} 2^n = \infty$.[3] In other words, a decision maker should be willing to pay any price to enter this game. Nevertheless, obviously some prices might be too high for a rational decision maker to pay for playing. Moreover, even with the rebuttal that any dealer could only offer finite money for another to play the St. Petersburg lottery, people still make choices in finite St. Petersburg game, which is inconsistent with expected utility theory (Cox, Sadiraj, and Vogt 2008). In fact, many people agree that "few of us would pay even \$25 to enter such a game" (Hacking 1980).

Table 2: The Ellsberg Paradox

	30 balls	60 balls	
	r: red	b: blue	g: green
c_1	\$100	\$0	\$0
c_2	\$0	\$100	\$0
c_3	\$100	\$0	\$100
c_4	\$0	\$100	\$100

Now, by using our model, based on the assumption of finite St. Petersburg game, suppose the net worth of Bill Gates in Forbes 2015 ($W = \$79.2$ Billion) is the total money that a potential player can offer, and a is the money that the loss aversion proneness decision maker is willing to pay. Then $L = \lfloor \log_2(W) \rfloor = 36$ is the maximum number of times the dealer can fully cover the bet, and the risky expected utility with RLR for playing the finite St. Petersburg game is:

$$L_r(c) = \sum_{n=1}^{36} \frac{1}{2^n} \times 2^n + 2\frac{W}{2^{L+1}} - (a-0) = 37.15 - a.$$

If a decision maker wants to play the game, by Definition 3, it means $37.15 - a \geq a$, and thus $a < 18.6$. Therefore, since the agent cannot even offer the net worth of Bill Gates money for such a game, it is no doubt that many people are willing to pay less money for playing the game. By considering the effect of worst case loss aversion, our model indeed gives an explanation for the St. Petersburg game.

The Ellsberg paradox is a well-known, long-standing one about ambiguity (Ellsberg 1961). Suppose in an urn containing 90 balls, 30 is red, and the rest are either blue or green, and a decision maker faces two pairs of decision problems, each involving a decision between two choices: c_1 and c_2, or c_3 and c_4. A ball is randomly picked up from the urn, and the return of selecting a ball for each choice is shown in Table 2. Ellsberg found that a very common pattern of human responses to these problems is: $c_1 \succ c_2$ and $c_4 \succ c_3$, which violates expected utility theory.

In the Ellsberg paradox, by Definition 4, the structure partition is $[S] = \{\{r\}, \{b, g\}\}$ with a chance of 1/3 for $\{r\}$ and a chance of $2/3$ for $\{b, g\}$. Then, by Definitions 2, 5 and 6,

[3]Here, although utility does not necessarily equal to money value and Bernoulli claims that the logarithmic utility function can handle this paradox in (Bernoulli 1954), many have found this response to the paradox unsatisfactory (Martin 2014).

Table 3: 50:51 Example

	50 balls		51 balls	
	e_1	e_2	e_3	e_4
c_1	\$80	\$80	\$40	\$40
c_2	\$80	\$40	\$80	\$40
c_3	\$120	\$80	\$40	\$0
c_4	\$120	\$40	\$80	\$0

Table 4: Reflection Example

	50 balls		51 balls	
	e_1	e_2	e_3	e_4
c_1	\$40	\$80	\$40	\$0
c_2	\$40	\$40	\$80	\$0
c_3	\$0	\$80	\$40	\$40
c_4	\$0	\$40	\$80	\$40

we have:

$$
\begin{aligned}
L_a(c_1) &= \frac{100}{3}-(0-0)=\frac{100}{3},\\
L_a(c_2) &= \frac{200}{3}-\frac{2}{3}\times(\frac{200}{3}-0)-(0-0)=\frac{200}{9},\\
L_a(c_3) &= 100-\frac{2}{3}\times(100-\frac{100}{3})-(0-0)=\frac{500}{9},\\
L_a(c_4) &= \frac{200}{3}-(0-0)=\frac{200}{3}.
\end{aligned}
$$

Thus, we have $c_1 \succ_a c_2$ and $c_4 \succ_a c_3$.

Since the Ellsberg paradox shows that a decision maker exhibits a preference for the choices with a deterministic probability distribution over the choices with undetermine probability distribution, it reveals a phenomenon, known as ambiguity aversion. And our method can resolve this type of paradox. Thus, our model indeed reflects the decision maker's ambiguity aversion nature by considering the influence of ambiguity loss aversion.

Recently, Machina (2009) has posed some questions regarding the ability of Choquet expected utility theory (Quiggin 2012) and some well-known ambiguity decision models to resolve variations of the Ellsberg paradox. And the examples of these questions are called the Machina paradox (Machina 2009).

The first example is shown in Table 3. In this 50:51 example, the pair of c_1 and c_2 and the pair of c_3 and c_4 differ only in whether they offer the higher prize of \$80 on event e_2 or event e_3. As argued by Machina (2009), a decision maker prefers c_1 to c_2, but the decision maker may feel that the tiny ambiguity difference between c_3 and c_4 does not offset c_4 objective advantage (a slight advantage due to that the 51st ball may yield \$80), and therefore prefers c_4 to c_3.

Using our method, by Definition 4, the structure partition is $[S] = \{\{e_1, e_2\}, \{e_3, e_4\}\}$ with a chance of $50/101$ for the set of $\{e_1, e_2\}$ and a chance of $51/101$ for the set of $\{e_3, e_4\}$. Then, by Definitions 2, 5 and 6, we can find $L_a(c_1) = 59.8$, $L_a(c_2) = 40$, $L_a(c_3) = 59.6$, and $L_a(c_4) = 59.8$. Thus, we have $c_1 \succ_a c_2$ and $c_4 \succ_a c_3$. Moreover, for c_1 and c_2, although c_2 has a slight advantage due to that the 51st ball may yield \$80, a decision maker still chooses c_1 because of ambiguity aversion. Whilst, for c_3 and c_4, since the ambiguity degree for both choices are the same, the slight advantage of c_4 influences the decision maker's choice. In other words, our method can consider the ambiguity aversion of the decision maker as well as the advantage of a higher utility for a decision making under ambiguity. So, our model covers well the 50:51 example in the Machina paradox.

Now, we consider the second type of the Machina paradox, called the reflection example in Table 4. In this example, since c_4 is an informationally symmetric left-right reflection of c_1 and c_3 is a left-right reflection of c_2, any decision maker who prefers c_1 to c_2 should have the "reflected" ranking that prefers c_4 to c_3. And if $c_2 \succ c_1$ then $c_3 \succ c_4$. The experimental analyses of L'Haridon and Placido (2008) found that over 90 percent of subjects expressed strict preference in the reflection problems, and that roughly 70 percent manifest the structure $c_1 \succ c_2$ and $c_4 \succ c_3$ or $c_2 \succ c_1$ and $c_3 \succ c_4$. And such a result cannot be explained well by most of ambiguity decision models (Baillon and Placido 2011).

Nonetheless, using our method, by Definition 4, the structure partition is $[S] = \{\{e_1, e_2\}, \{e_3, e_4\}\}$ with a chance of $1/2$ for $\{e_1, e_2\}$ and a chance of $1/2$ for $\{e_3, e_4\}$. Then, by Definitions 2, 5 and 6, we can find $L_a(c_1) = L_a(c_4) = 34.8$ and $L_a(c_2) = L_a(c_3) = 47.4$. Thus, $c_2 \succ_a c_1$ and $c_3 \succ_a c_4$. So, our method can also explain the reflection example.

Related Work

There are two strands of literature related to our analyses on the four well-known paradoxes. First, there are various generalised expected utility theories and ambiguity decision models that focus on some of the paradoxes. Generalised expected utility theories include prospect theory (PT) (Kahneman and Tversky 1990), regret theory (RT) (Bleichrodt and Wakker 2015; Zhang et al. 2016), rank-dependent expected utility (REU) (Quiggin 2012; Jeantet and Spanjaard 2011), cumulative prospect theory (CPT) (Tversky and Kahneman 1992; Barberis 2013; Wang, Wang, and Martnez 2017). Ambiguity decision models include maxmin expected utility (MEU) (Gilboa and Schmeidler 1989; Troffaes 2007), variational preferences (VP) (Maccheroni, Marinacci, and Rustichini 2006; De Marco and Romaniello 2015), α-maximin (αM) (Ghirardato, Maccheroni, and Marinacci 2004) and a smooth model of ambiguity aversion (SM) (Klibanoff, Marinacci, and Mukerji 2005). However, all of them cannot provide an explanation for all the paradoxes we analysed in this paper. In addition, our model is different from the most famous descriptive decision-making model (*i.e.*, prospect theory) in the following aspects: (i) they need some extra parameters, but we do not; and (ii) they cannot resolve St. Petersburg paradox, Ellsberg paradox and Machina paradox and even its extension version (*i.e.*, cumulative prospect theory) cannot resolve Machina paradox, either; but our model can resolve them all without setting any extra parameter.

Second, recently some decision models (Gul and Pesendorfer 2014) were proposed with the claim of resolving these paradoxes with some parameter variations. For example, the expected uncertainty utility (EUU) theory (Gul

Table 5: Models Comparison

	PT	RT	REU	CPT	MEU	VP	αM	SM	EUU	ours
A	+p	+p	+p	+p	N	$*$	N	$*$	+p	Y
S	N	N	$*$	+p	N	N	N	N	$*$	Y
E	N	+p	+p	+p	Y	+p	+p	+p	+p	Y
M	N	N	N	N	N	N	N	N	+P	Y

and Pesendorfer 2014) resolves the paradoxes based on the interval-valued expected utility and the utility indexes (a parameter variation) applied to the upper and lower bounds of the interval. Nevertheless, such a model has the following limitations: (i) it is unclear why a decision maker chooses different utility indexes in different decision problems; and (ii) since the utility indexes are obtained after the human selection of a given decision problem is known, it is unclear whether or not the utility indexes have the same predication power for any new problem.

Finally, Table 5 summarises the models comparison in solving four well-known paradoxes, where A stands for the Allais paradox, S stands for the St. Petersburg paradox, E stands for the Ellsberg paradox, M stands for the Machina paradox, $+p$ means the model can solve a paradox with some additional parameters, N means the model cannot solve a paradox, Y means the model can solve a paradox without any additional parameter, and $*$ means it is unclear whether or not the model can resolve a paradox.

Conclusion

This paper proposed a new decision model based on the human nature of loss aversion to address four well-known paradoxes (Allais and Hagen 2013; Ellsberg 1961; Joyce 2011; Machina 2009) at the same time without any additional parameter. More specifically, we distinguished two types of loss aversion: worst case loss aversion and ambiguity loss aversion in decision making under uncertainty, and proposed the formal definition of relative loss reduction in decision problems under risk or ambiguity. Then, we proved that the loss aversion in decision problems under risk or ambiguity can be defined by our unifying decision model with relative loss reduction. Moreover, we proved that some desirable properties can hold for our risky expected utility with relative loss reduction or ambiguous expected utility with relative loss reduction. Finally, we validated our models by resolving the four famous paradoxes and demonstrated that such a resolution without any additional parameter cannot be achieved by the existing decision models.

There are many possible extensions to our work. Perhaps the most interesting one is the axiomatisation and psychological experimental analyses of our method. Another tempting avenue is to extend our model for resolving some paradoxes in multi-criteria decision making and game theory. Finally, it is worth discussing the use of our method in some real-word applications such as health care, e-commerce, supply chain management, policy making, human-computer negotiation, security surveillance system and decision-theoretic planning.

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China under Grant Nos. 61772210, 61272066 and 61762016, the key project in universities in Guangdong Province of China under Grant No. 2016KZDXM024, the innovation project of postgraduate education in Guangdong Province of China under Grant No. 2016SFKC_13, the Natural Science Foundation of Guangdong Province of China under Grant Nos. 2016A030313231 and S2012030006242, Foundation for Young Talents in Higher Education of Guangdong under Grant No. 2016KQNCX030, the China Postdoctoral Science Foundation under Grant No. 2017M612688, and South China Normal University Young Teacher Research and Cultivation Fund Project under Grant No. 16KJ13.

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