On the Approximation of Nash Equilibria in Sparse Win-Lose Games

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Abstract

We show that the problem of finding an approximate Nash equilibrium with a polynomial precision is PPAD-hard even for two-player *sparse win-lose* games (i.e., games with $\{0, 1\}$-entries such that each row and column of the two $n \times n$ payoff matrices have at most $O(\log n)$ many ones). The proof is mainly based on a new class of prototype games called *Chasing Games*, which we think is of independent interest in understanding the complexity of Nash equilibrium.

Introduction

Game theory is a field that studies the conflict and cooperation between rational agents. For non-cooperative games, the concept of Nash equilibria (Nash 1951; 1950) captures the stable state of complex interactions between agents and has been used as a highly influential tool to analyze the behavior of selfish and rational players. Once each player plays according to a Nash equilibrium, one cannot change her strategy even if she knows strategies from others; in other words, no single player can gain a strictly higher payoff by deviating from a Nash equilibrium.

Nash equilibrium has attracted great attention from different communities including researchers from economics, biology and computer science due to its fundamental applications on various fields and its beautiful, deep mathematical structure. But given a game, how can we find one such equilibrium? Much effort has been devoted to the design of efficient algorithms for computing a Nash equilibrium based on mathematical programming and other methods (Garcia, Lemke, and Luethi 1973; Kuhn 1961; Lemke and J. T. Howson 1964; McKelvey and McLennan 1996; Shapley 1974; Wilson 1971). However, no polynomial-time algorithm is known after more than sixty years since Nash's first paper.

In the theoretical computer science community, the computational complexity of Nash equilibria has been studied intensively during the past decade. (Daskalakis, Goldberg, and Papadimitriou 2006) first showed that finding a Nash equilibrium in four-player game (4-Nash) is PPAD-hard, where PPAD is a complexity class introduced by (Papadimitriou 1994) to characterized total search problems based

on the parity argument such as Brouwer's fixed point theorem. Using the DGP framework, (Chen and Deng 2006) later showed that finding a Nash equilibrium remains PPAD-hard even in two-player games (or bimatrix games).

With significant progress on the complexity of Nash equilibria, people continued to distill the hard core of the hardness, and more complexity results were proved for two-player games with specific restrictions — What if most of the entries in payoff matrices are zeros? What if every entry is as simple as win-lose (0 or 1)? What if the precision is lower? (Chen, Teng, and Valiant 2007) proved that finding a Nash equilibrium in a win-lose game, where each entry of the two payoff matrices is either 0 or 1, is still PPAD-hard. Another result by (Chen, Deng, and Teng 2006) showed that in sparse games, in which each row and column contain at most 10 non-zero entries, approximating a Nash equilibrium is also PPAD-hard. Recently (Rubinstein 2016) proved that approximating a Nash equilibrium of two-player games with even constant precision needs $n^{\log^{1-o(1)} n}$ time, assuming some convinced hypothesis.

Since the first two results are both PPAD-hard, which is believed to be more difficult than the third one. However, we still cannot understand what the complexity of a sparse game or a win-lose game is. For sparse games, entries in the payoff matrices could be very complicated; for win-lose games, there are so many possible ways to construct one row in the matrices. One natural question arises from (Chen, Teng, and Valiant 2007) and (Chen, Deng, and Teng 2006): what is the complexity of finding a Nash equilibrium in a two-player game that is both sparse and simple? In this paper, we study this simple class of two-player games, showing that these two kinds of game have their own difficulties separatively.

Our results

We continue to explore the hard-core of the hardness of Nash equilibria in two-player games, by showing that the following problem is also PPAD-hard: Given a pair of $n \times n$ payoff matrices in which every entry is either 0 or 1 and each row and column contain at most $O(\log n)$ ones, we are asked to find an ϵ-approximate Nash equilibrium with a precision ϵ polynomially small in n, that is, correctly compute the equilibrium strategies of two players with a logarithmic number of 0-1bits. Note that for any k-sparse two-player game, a pair of the uniform distributions over all strategies gives us

a (k/n)-approximate Nash equilibrium. It is unlikely to use the base game used by previous work (Chen and Deng 2006; Chen, Deng, and Teng 2006; Daskalakis, Goldberg, and Papadimitriou 2006; Goldberg and Papadimitriou 2006) if we want to prove the hardness result for such a simple class of games. For this purpose we introduce a new class of games, named *Chasing Games*, which enables us to leverage the techniques used in (Chen, Teng, and Valiant 2007; Chen, Deng, and Teng 2006). Using such games, we also improve the result of (Chen, Deng, and Teng 2006) to the case where no entry is negative.

Related Work

Starting from the seminal paper (Daskalakis, Goldberg, and Papadimitriou 2006), the complexity of Nash equilibrium has be widely studied. Lots of works have been done using the similar framework (Chen and Deng 2005; Daskalakis and Papadimitriou 2005; Chen, Deng, and Teng 2009).

Our problem is closely related to the following two problems: complexity of Nash equilibria in sparse games (Chen, Deng, and Teng 2006) and win-lose games (Abbott, Kane, and Valiant 2005; Chen, Teng, and Valiant 2007). Sparse games consider the case where each row and column of the two payoff matrices contain at most 10 non-zero entries. (Chen, Deng, and Teng 2006) proved that polynomially approximating a Nash equilibrium in sparse games is PPAD-hard. They redesigned several gadgets used in (Chen, Deng, and Teng 2009) which proved that general two-player games are hard, such that they can go through all the argument and make the resulted game sparse simultaneously. In their work, each non-zero entry can be either negative or positive. It seems unlikely for us to encode the negative entry (i.e., representing a negative entry with several 0-1 payoff entries in our reduction). Note that we cannot shift the entries additively to make them positive, since this would result in a very dense matrix.

Each entry in the Win-lose games is either 0 or 1. (Chen, Teng, and Valiant 2007) showed that such simple games are also as hard as the general two-player games. The reduction heavily relies on that they can add as many ones as they want, which cannot guarantee the sparsity of the games.

For the algorithmic aspect, (Codenotti, Leoncini, and Resta 2006) and (Chen, Deng, and Teng 2006) gave polynomial-time algorithms to find a Nash equilibrium in win-lose games with at most two ones in each row and column. Extending their results to win-lose game with more ones in each row and column is a challenging problem. (Hermelin et al. 2013) gave an $l^{O(kl)} \cdot n^{O(1)}$-time algorithm for l-sparse game where the support size of Nash equilibrium is bounded by k. Note that if $k = O(n)$, this algorithm still requires exponential time.

Preliminaries

In this section, we will introduce the notations and necessary definitions.

Let Δ^n denote the set of all *probability vectors* in $\mathbb{R}^n$, that is, $\Delta^n := \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1, x_i \geq 0, \forall 1 \geq$

$i \geq n\}$. For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, let $\mathbf{A}_i$ denote the ith row of $\mathbf{A}$, $\mathbf{A}^j$ denote the jth column of $\mathbf{A}$. When we do some operation between a scalar a with vector (matrix) $\mathbf{x}$, we mean we operate the scalar a with each entry in $\mathbf{x}$, e.g., $(a\mathbf{x})_i = ax_i, \forall i \in [n]$, where $a \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^n$.

Given an integer $K > 0$, we say a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is K-*weak-scaled* if each entry in $\mathbf{A}$ can be represented as r/K where r is between 0 and K, and each column of $\mathbf{A}$ has one entry at least $1/6$. [1]

We begin with the definition of a bimatrix game.

Definition 1 (Bimatrix Game). *A bimatrix game $\mathcal{G}$ is defined by two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$, such that these two players have m and n actions to choose, respectively. If the first player chooses the ith action and the second player chooses the jth action, then their payoffs are $\mathbf{A}_{i,j}$ and $\mathbf{B}_{i,j}$, respectively.*

A *mixed strategy* of a player is a distribution over her choices. Given any $\mathbf{x} \in \Delta^m, \mathbf{y} \in \Delta^n$, a pair of mixed strategies $(\mathbf{x}, \mathbf{y})$ for a bimatrix game $\mathcal{G} = (\mathbf{A}, \mathbf{B})$, we define their expected payoffs are $\mathbf{x}^T \mathbf{A} \mathbf{y}$ and $\mathbf{x}^T \mathbf{B} \mathbf{y}$, respectively. A *Nash Equilibrium* is a pair of mixed strategies $(\mathbf{x}^*, \mathbf{y}^*)$ such that no single player can gain a strictly higher payoff by deviating from it. Formally, for any $\mathbf{x} \in \Delta^m$ and $\mathbf{y} \in \Delta^n$, we always have

$$\mathbf{x}^T \mathbf{A} \mathbf{y}^* \leq (\mathbf{x}^*)^T \mathbf{A} \mathbf{y}^*, \quad (\mathbf{x}^*)^T \mathbf{B} \mathbf{y} \leq (\mathbf{x}^*)^T \mathbf{B} \mathbf{y}^*.$$

The celebrated theorem of Nash (Nash 1950; 1951) shows the existence of Nash Equilibrium for any bimatrix game.

In this paper, we focus on the *approximate* Nash Equilibrium. We are interested in the following two kinds of approximations.

Definition 2 (ϵ-approximate Nash equilibrium). *Given $\epsilon > 0$, we say a pair of mixed strategy $(\mathbf{x}^*, \mathbf{y}^*)$ for game $\mathcal{G} = (\mathbf{A}, \mathbf{B})$ is ϵ-approximate Nash equilibrium if for any mixed strategies $\mathbf{x} \in \Delta^m$ and $\mathbf{y} \in \Delta^n$, we have*

$$\mathbf{x}^T \mathbf{A} \mathbf{y}^* \leq (\mathbf{x}^*)^T \mathbf{A} \mathbf{y}^* + \epsilon,$$

$$(\mathbf{x}^*)^T \mathbf{B} \mathbf{y} \leq (\mathbf{x}^*)^T \mathbf{B} \mathbf{y}^* + \epsilon.$$

Definition 3 (ϵ-well-supported Nash equilibrium). *We say a pair of mixed strategy $(\mathbf{x}^*, \mathbf{y}^*)$ for game $\mathcal{G} = (\mathbf{A}, \mathbf{B})$ is ϵ-well-supported Nash equilibrium if for any i, j, we have*

$$(\mathbf{x}^*)^T \mathbf{B}^i > (\mathbf{x}^*)^T \mathbf{B}^j + \epsilon \Rightarrow \mathbf{y}_j^* = 0,$$

$$\mathbf{A}_i \mathbf{y}^* > \mathbf{A}_j \mathbf{y}^* + \epsilon \Rightarrow \mathbf{x}_j^* = 0.$$

The first concept is the most common-used one, while the other is more convenient in the reductions since we can focus on the comparisons between any two pure strategy. (Chen, Deng, and Teng 2009) proved that these two definition is equivalent up to polynomial factors.

Lemma 1. *Given a bimatrix game $(\mathbf{A}, \mathbf{B})$ where $\mathbf{A}, \mathbf{B} \in [0, 1]^{n \times n}$, and $0 \leq \epsilon \leq 1$,*

- *each ϵ-well-supported Nash equilibrium is also an ϵ-approximate Nash equilibrium; and*

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[1] In (Chen, Teng, and Valiant 2007), they defined the notion K-*well-scaled* for a matrix if each entry is r/K for r should be between $K/2$ and K.

- *from any $\epsilon^2/8$-approximate Nash equilibrium $(\mathbf{u}, \mathbf{v})$, we can convert it to an ϵ-well-supported Nash equilibrium $(\mathbf{x}, \mathbf{y})$.*

Next we define the simple class of games we consider in this work, we call it Sparse Win-Lose games.

Definition 4. *Given a bimatrix game $\mathcal{G} = (\mathbf{A}, \mathbf{B})$, where $\mathbf{A}, \mathbf{B} \in \{0,1\}^{n \times n}$. We call $\mathcal{G}$ is* sparse win-lose *if every column and row of these two matrices have at most $O(\log n)$ ones.*

Review of the Former Reductions

Since the whole proof is a long and involved reduction based on previous work and mature framework, we decide to review two reduction related to our work in this section. The first one is the reduction in (Chen, Deng, and Teng 2006), from BROUWER^f to the problem of computing an n^{-6}-well-supported Nash equilibrium in a constant-sparse game, where BROUWER^f is a problem whose solution is proved to be existed by the fixed point theorem named Brouwer (Brouwer 1912), and it is known to be PPAD-complete (Chen and Deng 2009). The main idea is to make the circuit, the input of BROUWER^f sparse, that is, each node in the circuit can be used by at most two gates. Then they redesigned some of the gadgets in (Chen, Deng, and Teng 2009), such that they can show the same proofs and keep the resulting game sparse simultaneously.

Let $U = (C, 0^{3n})$ be an input instance of BROUWER^f, where C is a circuit. Let m be the smallest integer such that $2^m > \text{size}[C] > n$, where $\text{size}[C]$ is the number of gates plus the number of input and output variables in C. Given U, a bimatrix game $\mathcal{G}^U = (\mathbf{A}^U, \mathbf{B}^U)$ can be constructed in polynomial time, where $\mathbf{A}^U$ and $\mathbf{B}^U$ are $N \times N$ matrices, and $N = 2^{6m+1} = 2K$. The bimatrix game $\mathcal{G}^U$ has the following properties:

$\mathbf{P_0}$: Each row (column) of matrices $\mathbf{A}^U$ and $\mathbf{B}^U$ has at most 10 non-zero entries;

$\mathbf{P_1}$: $|a_{i,j}^U|, |b_{i,j}^U| \leq N^3$ for each i, j: $1 \leq i, j \leq N$;

$\mathbf{P_2}$: From every ϵ-well-supported Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of $\mathcal{G}^U$, where $\epsilon = 1/K^3$, we can use $(\mathbf{x}, \mathbf{y})$ to find a panchromatic simplex of circuit C in polynomial time, i.e., a solution of the problem BROUWER^f.

Since the notion of ϵ-approximate Nash equilibria will be confused[2] when scaling the matrices $\mathbf{A}$ and $\mathbf{B}$, one can normalize these two matrices to $\overline{\mathcal{G}^U} = (\overline{\mathbf{A}^U}, \overline{\mathbf{B}^U})$ where $\overline{\mathbf{A}^U} = \mathbf{A}^U/N^3$ and $\overline{\mathbf{B}^U} = \mathbf{B}^U/N^3$ such that $|\overline{a_{i,j}^U}|, |\overline{b_{i,j}^U}| \leq 1$ for each i, j: $1 \leq i, j \leq N$. By $\mathbf{P_2}$, the problem of finding an n^{-6}-well supported Nash equilibrium in a constant-sparse game is PPAD-hard.

The construction of $\mathcal{G}^U$ begins with a prototype game $\mathcal{G}^* = (\mathbf{A}^*, \mathbf{B}^*)$ called *Generalized Matching Pennies* games. The matrix $\mathbf{A}^*$ is a $K \times K$ block-diagonal matrix, where each block is a 2×2 matrix of all M's, $M = 2K^3 =$

2^{18m+1}, and $\mathbf{B}^* = -\mathbf{A}^*$. Chen et al. (Chen, Deng, and Teng 2009) showed that this simple game $\mathcal{G}^*$ has nice properties.

Lemma 2. *Let $(\mathbf{A}, \mathbf{B})$ be a game with $0 \leq \mathbf{A} - \mathbf{A}^*, \mathbf{B} - \mathbf{B}^* \leq 1$. Let $(\mathbf{x}, \mathbf{y})$ be a 1-well-supported Nash equilibrium of $(\mathbf{A}, \mathbf{B})$, then for each $k \in [K]$, it satisfies the following constraint $\mathcal{R}$:*

$$1/K - \epsilon \leq \mathbf{x}[2k-1] + \mathbf{x}[2k] \leq 1/K + \epsilon;$$

$$1/K - \epsilon \leq \mathbf{y}[2k-1] + \mathbf{y}[2k] \leq 1/K + \epsilon.$$

The game $\mathcal{G}^U = (\mathbf{A}^U, \mathbf{B}^U)$ is constructed by adding several carefully designed "gadget" games to $\mathcal{G}^*$, such that they can implement kinds of arithmetic and logic operations approximately, finally $\mathcal{G}^U$ can simulate the instance U of the problem BROUWER^f. Each gadget is defined by a 7-tuple $(G, v_1, v_2, v_3, v, c, w)$, where G is the gadget type, and others are the nodes specified by the circuit C of BROUWER^f, let's use a mapping $\mathcal{C}$ to label these nodes. For each gadget T, they define the "gadget" game $(\mathbf{M}[T], \mathbf{N}[T])$ as Figure 1.[3] Each gadget game satisfies some kind of constraint $\mathcal{R}[T]$, e.g., for $T = (G_+, v_1, v_2, v_3, v, c, w)$, the constraint is $\mathcal{R}[T] := [\mathbf{x}[v] = \min(\mathbf{x}[\mathcal{C}(v_1)] + \mathbf{x}[\mathcal{C}(v_2)], \mathbf{x}[\mathcal{C}(v)] + \mathbf{x}[\mathcal{C}(v)+1]) \pm \epsilon]$, one can see the full constraint set in (Chen, Deng, and Teng 2006; 2009). Then they constructed a collection of gadgets $\mathcal{T}$ of size at most K to build $\mathcal{G}^U = (\mathbf{A}^U, \mathbf{B}^U) := \text{BUILDGAME}(U, \mathcal{G}^*, \mathcal{T})$:

$$\mathbf{A}^U = \mathbf{A}^* + \sum_{T \in \mathcal{T}} \mathbf{M}[T] \text{ and } \mathbf{B}^U = \mathbf{B}^* + \sum_{T \in \mathcal{T}} \mathbf{N}[T],$$

such that for each pair of gadgets $T = (G, v_1, v_2, v_3, v, c, w), T' = (G', v_1', v_2', v_3', v', c', w') \in \mathcal{T}$, we have $v \neq v', w \neq w'$, we say such a collection of gadgets is *valid*. Chen et al. (Chen, Deng, and Teng 2009) proved the following lemma.

Lemma 3. *Given a valid set of gadgets $\mathcal{T}$, we have, for each gadget $T \in \mathcal{T}$, any ϵ-well-supported Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of $\mathcal{G}^U = \text{BUILDGAME}(U, \mathcal{G}^*, \mathcal{T})$ satisfies the constraint $\mathcal{R}[T]$.*

Besides, the game $\mathcal{G}^U = (\mathbf{A}^U, \mathbf{B}^U)$ satisfies the condition in Lemma 2, and by Lemma 3, we can prove that each ϵ-well-supported Nash equilibrium of $\mathcal{G}^U$ satisfies the following $|\mathcal{T}| + 1$ constraints $\{\mathcal{R}, \mathcal{R}_1, \ldots, \mathcal{R}_{|\mathcal{T}|}\}$, with more technical argument, one can check $\mathcal{G}^U$ satisfies the property $\mathbf{P_2}$, hence the reduction works.

The second reduction we want to mention is due to (Chen, Teng, and Valiant 2007). They gave a construction which can transform the family of hard bimatrix games constructed by (Chen, Deng, and Teng 2009) to win-lose games, i.e., two-player games with $\{0, 1\}$ entries. The main idea is to encode any non-negative integer between 0 and n using $O(\log n)$ bit in payoff matrices. We need the following lemma, whose proof is similar with Lemma 5.8 in (Chen, Teng, and Valiant 2007).

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[2] Given $c > 0$, an ϵ-approximate Nash equilibria $(\mathbf{x}, \mathbf{y})$ of game $\mathbf{A}, \mathbf{B}$, it will be $(c\epsilon)$-approximate Nash equilibria for the game $(c\mathbf{A}, c\mathbf{B})$.

[3] For $G_\wedge^*$, we modify two terms to $3/8$ ($1/3$ in the original paper) to make our proof more convenient while every result still holds.

Set $\mathbf{M}[T] = (M_{i,j}) = \mathbf{N}[T] = (N_{i,j}) = 0$, $k = \mathcal{C}(v)$, $k_1 = \mathcal{C}(v_1)$, $k_2 = \mathcal{C}(v_2)$, $k_3 = \mathcal{C}(v_3)$ and $t = \mathcal{C}(w)$

$$G_+ : \begin{cases} M_{2k-1,2t-1} = M_{2k,2t} = 1 \\ N_{2k_1-1,2t-1} = N_{2k_2-1,2t-1} = N_{2k-1,2t} = 1 \end{cases}$$

$$G_- : \begin{cases} M_{2k-1,2t-1} = M_{2k,2t} = 1 \\ N_{2k_1-1,2t-1} = N_{2k_2-1,2t} = N_{2k-1,2t} = 1 \end{cases}$$

$$G_= : \begin{cases} M_{2k-1,2t-1} = M_{2k,2t} = 1 \\ N_{2k_1-1,2t-1} = N_{2k-1,2t} = 1 \end{cases}$$

$$G_< : \begin{cases} M_{2k-1,2t} = M_{2k,2t-1} = 1 \\ N_{2k_1-1,2t-1} = N_{2k_2-1,2t} = 1 \end{cases}$$

$$G_{\times\zeta} : \begin{cases} M_{2k-1,2t-1} = M_{2k,2t} = 1 \\ N_{2k-1,2t} = 1, N_{2k_1-1,2t-1} = c \end{cases}$$

$$G_\neg : \begin{cases} M_{2k-1,2t} = M_{2k,2t-1} = 1 \\ N_{2k_1-1,2t-1} = N_{2k_1,2t} = 1 \end{cases}$$

$$G_\zeta^* : \begin{cases} M_{2k-1,2t} = M_{2k,2t-1} = 1 \\ N_{2k-1,2t-1} = 1/2, N_{2k_3-1,2t} = Kc \end{cases}$$

$$G_\wedge^* : \begin{cases} M_{2k-1,2t-1} = M_{2k,2t} = N_{2k_3-1,2t} = 1 \\ N_{2k_1-1,2t-1} = N_{2k_2-1,2t-1} = 3/8 \end{cases}$$

$$G_\vee^* : \begin{cases} M_{2k-1,2t-1} = M_{2k,2t} = 1 \\ N_{2k_1-1,2t-1} = N_{2k_2-1,2t-1} = N_{2k_3-1,2t} = 1 \end{cases}$$

$$G_{B=} : \begin{cases} M_{2k-1,2t-1} = M_{2k,2t} = 1 \\ N_{2k_1-1,2t-1} = N_{2k_1,2t} = 1 \end{cases}$$

$$G_H : \begin{cases} M_{2k-1,2t} = M_{2k,2t-1} = 1 \\ N_{2k-1,2t-1} = N_{2k,2t} = 1 \end{cases}$$

Figure 1: Construction of "Gadget" game $(\mathbf{M}[T], N[T])$, where $T = (G, v_1, v_2, v_3, v, c, w)$

Lemma 4. *Let $H = (\mathbf{A}, \mathbf{B})$ be a normalized bimatrix game and $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices. Both $\mathbf{A}$ and $\mathbf{B}^T$ are K-weak-scaled, where $K = 3(2^k - 1) \leq n^{19}$. We construct $H' = (\mathbf{A}', \mathbf{B}')$ as follows:*

$$\mathbf{A}' = \begin{pmatrix} \mathbf{S} & \mathbf{I} \\ \mathbf{R} & \mathbf{0} \end{pmatrix}, \mathbf{B}' = \begin{pmatrix} 1-\mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix},$$

where

- $\mathbf{S}$ *is an $n \times n$ block-diagonal matrix, in which each block is a $3k \times 3k$ 0-1 matrix, denoted by $\mathbf{S}^k$, defined as follows:*

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\mathbf{S}^k = \begin{pmatrix} \mathbf{U} & \mathbf{U} & \cdots & \mathbf{U} & \mathbf{V} \\ \mathbf{U} & \mathbf{U} & \cdots & \mathbf{V} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{U} & \mathbf{V} & \cdots & 0 & 0 \\ \mathbf{V} & 0 & \cdots & 0 & 0 \end{pmatrix};$$

$1 - \mathbf{S}$ *is also an $n \times n$ block-diagonal matrix, in which each block is a $3k \times 3k$ 0-1 matrix, denoted by $1 - \mathbf{S}^k$. The game $G_k = (\mathbf{S}^k, 1 - \mathbf{S}^k)$ is called* generator game

in (Abbott, Kane, and Valiant 2005), which has a unique Nash equilibrium $(\mathbf{r}, \mathbf{c})$ where

$$\mathbf{r} = \mathbf{c} = \frac{\left(2^{k-1}, 2^{k-1}, 2^{k-1}, \ldots, 4, 4, 4, 2, 2, 2, 1, 1, 1\right)^T}{3(2^k - 1)};$$

- $\mathbf{R}$ *is a $n \times n$ block matrix, each block is a row vector of length $3k$, such that $\mathbf{R}_{i,j} \cdot \mathbf{r} = \mathbf{A}_{i,j}$;*
- $\mathbf{I}$ *is a $n \times n$ block-diagonal matrix, each block is an all-one column vector of length $3k$.*

such that

1. *the matrices $\mathbf{A}'$ and $\mathbf{B}'$ are $n(3k+1) \times n(3k+1)$ matrices;*
2. $\mathbf{A}'$ *is a 0-1 matrix and each column has a non-zero entry;*
3. $\mathbf{B}'$ *has entries either 0, 1 or from $\mathbf{B}$.*

From any ϵ/n^{43}-well-supported Nash equilibrium $(\mathbf{x}', \mathbf{y}')$ of H', we can get an ϵ-well-supported Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of H.

Main Result

In this section, we first introduce a new class of prototype games named *Chasing Games*, and then use this kind of games to prove that approximating any Nash equilibrium with polynomial precision in sparse win-lose games is also PPAD-hard.

Chasing Games

Due to the simple structure and nice property, the GMP (Generalized Matching Pennies) games are widely used to prove PPAD-hardness results of Nash equilibria (Goldberg and Papadimitriou 2006; Chen, Deng, and Teng 2009; Daskalakis, Goldberg, and Papadimitriou 2006; Chen, Deng, and Teng 2006). However, we consider the sparse case, it seems impossible to use GMP game as the base game in our reduction again. Even though GMP games are sparse, the matrix $\mathbf{B}$ is negative, we don't know how to encode negative number using Lemma 4. We can also shift the entries in $\mathbf{B}$ to be positive to avoid the above issue, but it makes $\mathbf{B}$ so dense. We need a class of games which has the similar property as GMP but has no negative entry. We define our Chasing game as follows.

Definition 5 (Chasing Game). *Given an integer $n > 0$, chasing game $\mathcal{J}$ is defined by two matrices $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{2K \times 2K}$:*

$$\mathbb{A} = \begin{pmatrix} M & M & 0 & 0 & \cdots & 0 & 0 \\ M & M & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & M & M & \cdots & 0 & 0 \\ 0 & 0 & M & M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & M & M \\ 0 & 0 & 0 & 0 & \cdots & M & M \end{pmatrix},$$

$$\mathbb{B} = \left(\mathbb{A}^{2K-1}, \mathbb{A}^{2K}, \mathbb{A}^1, \mathbb{A}^2, \ldots, \mathbb{A}^{2K-3}, \mathbb{A}^{2K-2}\right),$$

where $M = 2K^3 = 2^{18m+1}$ and m is the smallest integer such that $2^m > n$.

First, let's prove the same statement as Lemma 2 for Chasing game, i.e., every approximate Nash equilibrium of Chasing games is almost uniform over all strategies.

Lemma 5. *Let* $(\mathbf{A}, \mathbf{B})$ *be a game with* $0 \le \mathbf{A} - \mathbb{A}, \mathbf{B} - \mathbb{B} \le 1$. *If* $(\mathbf{x}, \mathbf{y})$ *is a 1-well-supported Nash equilibrium of* $\mathbf{A}, \mathbf{B}$, *for any* $k \in [K]$, *we have*

$$1/K - 1/K^3 \le \mathbf{x}[2k-1] + \mathbf{x}[2k] \le 1/K + 1/K^3;$$
$$1/K - 1/K^3 \le \mathbf{y}[2k-1] + \mathbf{y}[2k] \le 1/K + 1/K^3.$$

Proof. Here we just prove the first inequality, the second one holds with the similar argument. Let's denote $\mathbf{x}[k]^+ := \mathbf{x}[2k-1] + \mathbf{x}[2k]$, where $k \in [K]$, the similar with $\mathbf{y}[k]^+$.

Now we prove that $|\mathbf{x}[k]^+ - 1/K| \le 1/K^3$ for any $k \in [K]$. Assuming that the statement is not true, that is, there exists some $i \in [K]$, such that $|\mathbf{x}[i]^+ - 1/K| > 1/K^3$. Without the loss of generality, we assume that $\mathbf{x}[i]^+ > 1/K + 1/K^3$. Since $\sum_{k=1}^{K} x[k]^+ = 1$, there must exist some $j \in [K]$, such that $\mathbf{x}[j]^+ < 1/K$, then we have

$$\mathbf{x}[i]^+ - \mathbf{x}[j]^+ > 1/K^3. \tag{1}$$

By definition, for any $k \in [K]$, the $2k - 1$st and $2k$th entries on the rows $\mathbf{A}_{2k-1}$ and $\mathbf{A}_{2k}$ are in $[M, M+1]$, while others are in $[0, 1]$. So we have

$$M\mathbf{y}[k]^+ \le \mathbf{A}_{2k-1}\mathbf{y}, \mathbf{A}_{2k}\mathbf{y} \le M\mathbf{y}[k]^+ + 1, \forall \mathbf{y} \in \Delta^{2K}. \tag{2}$$

Similarly, for any $k \in [K]$, the $2k - 1$st and $2k$th entries on the columns $\mathbf{B}^{2k+1}$ and $\mathbf{B}^{2k+2}$ are in $[M, M+1]$ (if $k = K$, the columns are the B_1 and B_2), while others are in $[0, 1]$, we have

$$M\mathbf{x}[k]^+ \le \mathbf{x}^T\mathbf{B}^{2k+1}, \mathbf{x}^T\mathbf{B}^{2k+2} \le M\mathbf{x}[k]^+ + 1, \forall \mathbf{x} \in \Delta^{2K}. \tag{3}$$

Combining these three equations, we will prove that the mixed strategy $\mathbf{y}$ is a zero vector, hence our assumption is wrong and the lemma holds.

By Equation 3, we consider the difference of expected payoff between columns $\mathbf{B}^{2i+1}$ and $\mathbf{B}^{2j+1}, \mathbf{B}^{2j+2}$:

$$\mathbf{x}^T\mathbf{B}^{2i+1} - \max\{\mathbf{x}^T\mathbf{B}^{2j+1}, \mathbf{x}^T\mathbf{B}^{2j+2}\}$$
$$\ge M\mathbf{x}[i]^+ - (M\mathbf{x}[j]^+ + 1) > M/K^3 - 1 > 1.$$

Since $\mathbf{x}, \mathbf{y}$ is 1-well-supported Nash equilibrium, we have $\mathbf{y}[j+1]^+ = 0$, and there exists $j' \in [K]$ such that $\mathbf{y}[j']^+ > 1/K$. By Equation 2 and the similar argument, we have

$$\mathbf{A}_{2j'-1}\mathbf{y} - \max\{\mathbf{A}_{2j+1}\mathbf{y}, \mathbf{A}_{2j+2}\mathbf{y}\}$$
$$\ge M\mathbf{y}[j']^+ - (M\mathbf{y}[j+1]^+ + 1) > M/K - 1 \gg 1.$$

Hence we have $\mathbf{x}[j+1]^+ = 0$. Repeat the proof above, we can see that $\mathbf{x}$ and $\mathbf{y}$ are both zero vectors, a contradiction. □

For a valid collection of gadgets, by the definition of valid gadgets, each gadget game modifies different rows or columns in the base game, and it doesn't care about what base game you use. Hence the proof of following lemma for Chasing games is similar with Lemma 3.

Lemma 6. *Given a valid set of gadgets* $\mathcal{T}$, *we have, for each gadget* $T \in \mathcal{T}$, *any* ϵ-*well-supported Nash equilibrium* $(\mathbf{x}, \mathbf{y})$ *of* $\mathcal{G}^U = \textsc{BuildGame}(U, \mathcal{J}, \mathcal{T})$ *satisfies the constraint* $\mathcal{R}[T]$.

Reduction and its correctness

In this section, we will prove the main result of our paper, approximating a Nash equilibrium in sparse win-lose game is PPAD-hard. Firstly, we improve the result in (Chen, Deng, and Teng 2006), showing that even approximating Nash equilibrium in non-negative constant-sparse games is also hard, by replacing GMP games with our chasing games. Then we transform the non-negative constant-sparse game to sparse win-lose game by modifying a construction in (Chen, Teng, and Valiant 2007), and build the connection between the solutions of these two games. Now we are ready to prove the hardness result for non-negative constant-sparse games.

Theorem 1. *Finding an* n^{-6}-*well-supported Nash equilibrium in non-negative constant-sparse game of dimension* $n \times n$ *is PPAD-hard.*

Furthermore, the numerator of each entry in the resulting game can be represented as one 0-1 bits of length $O(\log n)$ *with only constant ones.*

Proof. We use Chasing game as the prototype game of the reduction and use the same construction in (Chen, Deng, and Teng 2006). That is, for the same valid collection of gadgets $\mathcal{T}$ as (Chen, Deng, and Teng 2006), we build a game $H^U = (\mathbf{A}^U, \mathbf{B}^U) = \textsc{BuildGame}(U, \mathcal{J}, \mathcal{T})$. By Lemma 5 and Lemma 6, with the similar argument, one can prove the first statement.

It is easy to see that each entry of chasing game $\mathcal{J}$ will be changed at most once, since for a valid collection of gadget games, different gadget game occupies different columns or rows. From Figure 1, we can only add one of the following numbers $\{1, c, Kc', 1/2, 3/8\}$ to matrices in $\mathcal{J}$, and one can check that c and Kc' must be one of $\{1/2, 1/4, 1/8\}$ (cf. Fig 4. (Chen, Deng, and Teng 2009)). Hence each entry of the payoff matrices in $\mathcal{J}$ should be the sum of constant (at most two) number of terms in $\{M, 1, 1/2, 3/8, 1/4, 1/8\}$, the second statement holds. □

Next we will focus on the second phase, reducing from non-negative constant-sparse game to sparse win-lose game. We apply Lemma 4, which is modified from Lemma 5.8 in (Chen, Teng, and Valiant 2007). Chen et al. proved that approximating equilibrium in win-lose game is PPAD-hard by reducing from hard instances of two-player game in (Chen, Deng, and Teng 2009) to win-lose games.

The main idea of Lemma 4 is to embed the payoff matrices in an instance of hard games, $\mathbf{A}, \mathbf{B}$ into a 0-1 matrix and a matrix containing either $\{0, 1\}$'s or entries in $\mathbf{B}$, respectively. Recall that $\mathbf{A}$ is K-weak-scaled, one can encode each entry in $\mathbf{A}$ with a 0-1 string of length $\log K$. Then we exchange the roles of $\mathbf{A}$ and $\mathbf{B}$ to make $\mathbf{B}$ a 0-1 matrix. To reduce the dimension of our matrix, we will prove the following lemma by using Lemma 4 twice with different parameters.

Lemma 7. *There exists a pair of polynomial-time computable functions* f, g *such that given an* $n \times n$ *bimatrix game* $H = (\mathbf{A}, \mathbf{B})$ *and an integer* $K = 3(2^k - 1) \le n^{19}$ *such that* $\mathbf{A}$ *and* $\mathbf{B}^T$ *are both* K-*weak-scaled,* $f(H)$ *is a sparse win-lose game* $H' = (\mathbf{A}', \mathbf{B}')$ *of dimension*

$\Theta(nk)$, and for every ϵ/n^{87}-well-supported Nash equilibrium $(\mathbf{x}', \mathbf{y}')$ of game H', where $\epsilon \leq 1$, $(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}', \mathbf{y}')$ is an ϵ-well-supported Nash equilibrium of H.

Proof. We multiply each entry in the instance of hard games in Theorem 1 by $\frac{8}{3(2^{18m+5}-1)}$ such that each entry is between 0 and 1. Recall that the chasing game has non-zero entry $M = 2^{18m+1}$ in each column and row where m is the smallest integer such that $2^m > n$, we have each column of the matrix $\mathbf{A}$ has an entry $\frac{8M}{3(2^{18m+5}-1)} > 1/6$, hence the matrix $\mathbf{A}$ is K-weak-scaled, where $K = 3(2^k - 1) = 3(2^{18m+5} - 1) \leq n^{19}$. The similar argument with $\mathbf{B}^T$. So now we have $\mathbf{A}$ and $\mathbf{B}^T$ are both K-weak-scaled. We first apply Lemma 4 to construct a game $H' = (\mathbf{A}', \mathbf{B}')$ where $\mathbf{A}'$ and $\mathbf{B}'$ satisfy the properties in Lemma 4:

$$\mathbf{A}' = \begin{pmatrix} \mathbf{S} & \mathbf{I} \\ \mathbf{0} & \mathbf{R} \end{pmatrix}, \mathbf{B}' = \begin{pmatrix} 1 - \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}.$$

We can see that many rows in the upper blocks of $\mathbf{A}'$ contain $O(k) = O(\log n)$ ones while rows in the lower blocks contain only constant ones from Theorem 1. The next step is to apply Lemma 4 again to (B'^T, A'^T), to make the matrix B' to be a 0-1 matrix. One can see that many rows of matrix $\mathbf{B}'$ contains $O(\log n)$ ones in the $(1 - \mathbf{S})$ part, if we encode each of them with $\log n$ bits by the same construction, we can only get the hardness result for the game with at least $O(\log^2 n)$ ones in a column.

A crucial observation is that we can encode the entry which is 1 with only constant bits! The idea is to deal with $(1 - \mathbf{S})$ part and $\mathbf{B}$ part separately, we consider the sub-matrix $(1 - \mathbf{S})$ to be $K_1 = 3(2^2 - 1)$-weak-scaled while the $\mathbf{B}$ part to be $K_2 = 3(2^k - 1)$-weak-scaled. Since the analysis of Lemma 4 only considers each individual block separately, the properties of resulted game still hold in our setting. We set $n'' = n(3k + 1), K_1 = 9, K_2 = 3(2^k - 1)$, and game $H'' = (\mathbf{B}'^T, \mathbf{A}'^T)$. One can check that for any ϵ-well-supported Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of game H'', we have that $(\mathbf{y}, \mathbf{x})$ is an ϵ-well-supported Nash equilibrium of game H'. Now we apply the construction in Lemma 4 again to yield the game $H''' = (\mathbf{A}''', \mathbf{B}''')$ as follows:

$$\mathbf{A}''' = \begin{pmatrix} \mathbf{S} & \mathbf{I} \\ \mathbf{0} & \mathbf{R}_{\mathbf{B}'^T} \end{pmatrix}, \mathbf{B}''' = \begin{pmatrix} 1 - \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}'^T \end{pmatrix},$$

where $\mathbf{R}_{\mathbf{B}'^T}$ is used to encode the matrix $\mathbf{B}'^T$ by Lemma 4 and $n''' = 3kn \cdot 3 \cdot 2 + 3kn = \Theta(nk)$. As a by-product, we also reduce the dimension of the resulting game which was $\Theta(nk^2)$ in (Chen, Teng, and Valiant 2007). For the sparsity, it is easy to check that each row and column still have at most $O(\log n)$ ones.

Given any ϵ/n^{87}-well-supported Nash equilibrium $(\mathbf{x}''', \mathbf{y}''')$ of game H''', we can find $(\mathbf{x}'', \mathbf{y}'')$ is ϵ/n^{43}-well-supported Nash equilibrium of game H'' since $k = O(\log n)$ and

$$\frac{\epsilon}{n^{43}} \cdot \frac{1}{(n'')^{43}} = \frac{\epsilon}{n^{43}} \cdot \frac{1}{(n(3k+1))^{43}} > \frac{\epsilon}{n^{87}}.$$

So $(\mathbf{x}', \mathbf{y}') = (\mathbf{y}'', \mathbf{x}'')$ is an ϵ/n^{43}-well-supported Nash equilibrium of H' by our construction. By Lemma 4 again, we can get a ϵ-well-supported Nash equilibrium of H. $\square$

Combining Theorem 1, Lemma 7 and Lemma 1, we can prove our main theorem.

Theorem 2. *There exists a constant $c > 0$, such that find a n^{-c}-approximate Nash equilibrium in $n \times n$ sparse win-lose game is PPAD-hard.*

Remark. *In (Chen, Teng, and Valiant 2007), they proved the same hardness result for win-lose games with any constant $c > 0$. This cannot be true for sparse win-lose games, since for any game with only $\log n$ non-zero entries in each row and column, we can use two uniform distributions over all the strategies from each player to yield a $\log n/n$-well-supported Nash equilibrium.*

Conclusion

In this paper, we introduce a new class of two-player games named *Chasing games*. Using Chasing games we can prove approximating win-lose game with at most logarithmic ones entries in each column and row of bimatrix game, is also PPAD-hard. We strongly believe that this new class of games can be of independent interest, it can be used to prove new hardness result in Nash computation.

It is worth pointing out that our initial goal is to prove the same result for constant-sparse win-lose games in which each row and column have constant many ones. Also note that we can encode any entry of the hard instance in Theorem 1 with only constant ones using our techniques. However, some of the rows and columns in the game we construct do contain $\log n$ many ones. We conjecture that the same result holds for constant-sparse win-lose games but new insights and techniques are needed for understanding structures of sparse win-lose games.

Acknowledgments

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