Universal and Tight Online Algorithms for Generalized-Mean Welfare

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Abstract

We study fair and efficient allocation of divisible goods, in an online manner, among n agents. The goods arrive online in a sequence of T time periods. The agents' values for a good are revealed only after its arrival, and the online algorithm needs to fractionally allocate the good, immediately and irrevocably, among the agents. Towards a unifying treatment of fairness and economic efficiency objectives, we develop an algorithmic framework for finding online allocations to maximize the generalized mean of the values received by the agents. In particular, working with the assumption that each agent's value for the grand bundle of goods is appropriately scaled, we address online maximization of p-mean welfare. Parameterized by an exponent term $p \in (-\infty, 1]$, these means encapsulate a range of welfare functions, including social welfare ($p = 1$), egalitarian welfare ($p \to -\infty$), and Nash social welfare ($p \to 0$).

We present a simple algorithmic template that takes a threshold as input and, with judicious choices for this threshold, leads to both universal and tailored competitive guarantees. First, we show that one can compute online a *single allocation* that $O(\sqrt{n} \log n)$-approximates the optimal p-mean welfare for all $p \leq 1$. The existence of such a universal allocation is interesting in and of itself. Moreover, this universal guarantee achieves essentially tight competitive ratios for specific values of p.

Next, we obtain improved competitive ratios for different ranges of p by executing our algorithm with p-specific thresholds, e.g., we provide $O(\log^3 n)$-competitive ratio for all $p \in (\frac{-1}{\log 2n}, 1)$.

We complement our positive results by establishing lower bounds to show that our guarantees are essentially tight for a wide range of the exponent parameter.

1 Introduction

Resource-allocation settings are ubiquitous and often require the assignment of resources (goods) that arrive over time. In particular, these settings require that each good gets allocated (among the participating agents) as it arrives, and one needs to make these allocative decisions without knowing the values of the pending goods. Consider, as an example, a food bank that distributes food donations (essentially a divisible good) every day among soup kitchens (agents) (Prendergast 2017; Aleksandrov et al. 2015). Here, the perishable nature of the good (food) mandates online allocations, and supply (and demand) variability leads to limited information about the future. The online model is also applicable in scheduling contexts wherein computational resources, which become available over time, have to be shared among users (Blazewicz et al. 2019; Pinedo 2012; Leung 2004).

In such settings—and resource-allocation contexts, in general—economic efficiency and fairness are fundamental objectives. Motivated by these considerations and application domains, such as the ones mentioned above, a growing body of work in recent years has been directed towards the study of online fair division (Aleksandrov and Walsh 2020). The current paper contributes to this thread of research with a welfarist perspective.

Specifically, we provide a unified treatment of fairness and efficiency objectives by developing an online algorithm for finding allocations that maximize the *generalized mean* of the values achieved by the agents. Formally, for exponent parameter $p \in \mathbb{R}$, the pth generalized mean of n nonnegative values $\nu_1, \nu_2, \ldots, \nu_n$ is defined as $\left(\frac{1}{n} \sum_{i=1}^{n} \nu_i^p\right)^{1/p}$. As a family of objective functions (parameterized by p), generalized means encapsulate well-studied measures of economic efficiency as well as fairness: the $p = 1$ case corresponds to average social welfare (arithmetic mean), which is a standard measure of economic efficiency. At the other end of the spectrum with $p \to -\infty$, the pth generalized mean corresponds to egalitarian welfare (the minimum value across the agents), a fundamental measure of fairness. Furthermore, when p tends to zero, the generalized mean reduces to Nash social welfare (specifically, the geometric mean)–a prominent objective which achieves a balance between the extremes of social and egalitarian welfare. Notably, p-means, with $p \in (-\infty, 1]$, exactly constitute a family of functions characterized by a set of natural, fairness axioms, including the Pigou-Dalton principle (Moulin 2004). Therefore, p-mean welfare, with $p \leq 1$, provides us with an important and axiomatically justified family of objectives.

The current work focuses on the online allocation of divisible goods, i.e., goods that can be assigned fractionally among the n participating agents. The divisible goods arrive online, one by one, and upon the arrival of each good t, every agent i reports her (nonnegative) value, v_i^t, for the good.

At this point the online algorithm fractionally distributes the good t among the agents–if agent i receives $x_i^t \in [0,1]$ fraction of the good, then her value increases by $x_i^t \, v_i^t$. We have T goods overall and after the goods have been allocated, each agent i achieves a value $\sum_{t=1}^{T} x_i^t v_i^t$. Considering exponent parameters $p \leq 1$, the online algorithm's objective is to compute allocations that (approximately) maximize the p-mean of the agents' values. Recall that the performance of online algorithms is established in terms of their competitive ratio; in the current context, it is the worst case ratio (over all instances) between the optimal (offline) p-mean welfare and the welfare of the allocation computed online.

Along with this standard model and performance metric, we work with a scaling assumption utilized in prior work in online fair division (see, e.g., (Gkatzelis, Psomas, and Tan 2020), (Bogomolnaia, Moulin, and Sandomirskiy 2019), (Banerjee et al. 2021)); in particular, we assume throughout that, for every agent, the cumulative value of the entire set of goods is equal to one. Note that if the agents' valuations are arbitrarily scaled, then a sub-linear competitive ratio is not possible, even for specific values of the parameter $p < 1$; for unscaled valuations, Banerjee et al. (2021) provide an example that rules out a sub-linear competitive ratio specifically for Nash social welfare. To circumvent such overly pessimistic negative results, prior work has worked with this scaling assumption ($\sum_{t=1}^{T} v_i^t = 1$ for all agents i) and we conform to it as well. It is relevant to note that Banerjee et al. (2021) view this assumption in the framework of *algorithms with predictions*; see (Mitzenmacher and Vassilvitskii 2021) for a textbook treatment of this topic. In this framework, one assumes that the algorithm has a priori (side) information about each agent's value for the grand bundle.[1] Another way to realize this scaling is via scrips: upfront, each agent receives scrips (tokens) of overall value one and can distribute the tokens online to indicate her values over the goods. Note that the widely-used platform `spliddit.org` (Goldman and Procaccia 2015) asks for the valuations to be entered with a scaling in place, albeit in an offline manner. Overall, subject to the above-mentioned scaling, our results hold in the adversarial model wherein the value of each good t can be set by an adaptive adversary based upon, say, the fractional assignments till round $(t-1)$.

Related Work. Fair division has been extensively studied for over seven decades (Brams and Taylor 1996; Brandt et al. 2016; Moulin 2004). Since a detailed discussion on (offline) algorithms for fair division is beyond the scope of the current work, we will primarily focus on prior results on online algorithms and divisible goods. A work closely related to ours is that of Banerjee et al. (2021), who develop an $O(\log n)$-competitive online algorithm for maximizing Nash social welfare over divisible goods. Furthermore, Banerjee et al. provide an intricate lower bound showing that this competitive ratio is the best possible (up to an absolute constant) for Nash social welfare. The current work obtains an $O(\log^3 n)$ competitive ratio for Nash social welfare. Our algorithmic template, however, spans all $p \leq 1$, and provides tight—up to poly-log factor—competitive guarantees for a wide range of the exponent parameter. In terms of algorithm design, Banerjee et al. (2021) utilize the primal-dual method to obtain a competitive guarantee. Applications of this design schema, in related online settings, include the work of Devanur and Jain (2012) and Azar et al. (2010). By contrast, we develop a distinctive charging argument to design a single algorithmic template for all $p \leq 1$.

Bounding envy is another well-studied goal in the literature on online fair division; see, e.g., (Gkatzelis, Psomas, and Tan 2020; Benade et al. 2018; Bogomolnaia, Moulin, and Sandomirskiy 2019; Zeng and Psomas 2020), and (Aleksandrov and Walsh 2020) for a survey. Complementary to these results, we address p-mean welfare.

The current divisible-goods model captures machine scheduling with splittable jobs (Jansen et al. 2021; Correa et al. 2015; Epstein and Stee 2006). In this setup one needs to schedule T jobs among n machines, and each job can be split into multiple parts that get assigned to different machines. In contrast to p-mean welfare maximization, the focus of these scheduling results is on makespan minimization.

1.1 Our Results.

In this paper we develop both upper bounds and lower bounds for online welfare maximization.

Upper Bounds. Our online algorithm (see Section 3) works with a given threshold Φ. Setting this threshold judiciously enables us to obtain both expansive and tailored competitive guarantees.

First, we prove that a particular choice of the threshold (specifically $\Phi = 8\sqrt{n}\log(2n)$) leads to an online algorithm that achieves a universal competitive ratio of $O\left(\sqrt{n}\ \log n\right)$ for p-mean welfare maximization, simultaneously for all $p \leq 1$; recall that n denotes the total number of agents.

Theorem 1. *For the p-mean welfare maximization problem—with divisible goods and scaled valuations—one can compute online a single allocation that $O\left(\sqrt{n}\ \log n\right)$ approximates the optimal p-mean welfare, simultaneously for all $p \leq 1$.*

Theorem 1 is established in Section 4. This theorem in fact provides a novel guarantee specifically for egalitarian welfare ($p = -\infty$). Also, we note that Banerjee et al. (2021) prove—via a direct example—that, for egalitarian-welfare maximization and any constant $\varepsilon > 0$, there does not exist an online algorithm with competitive ratio $n^{1/2-\varepsilon}$. Hence, this lower bound ensures that, for egalitarian welfare, the competitive ratio of Theorem 1 is tight, up to a log term. Moreover, the guaranteed existence of a single allocation that achieves a nontrivial p-mean welfare guarantee, simultaneously for all $p \leq 1$, is interesting in its own right.

We also obtain improved guarantees for different ranges of p. In particular, we show that executing our algorithm with p-specific thresholds Φ leads to improved competitive ratios for a wide range of the exponent parameter $p \leq 1$. Our upper bounds are listed in Table 1. Interestingly, while

[1] The result of Banerjee et al. (2021) is robust to prediction errors. Extending the current work along these lines is an interesting direction of future work.

Range of p	Algorithm threshold Φ	Upper Bound	Lower Bound
Egalitarian Welfare ($p = -\infty$)	$8\sqrt{n}\log(2n)$	$O(\sqrt{n}\log n)$	$\Omega\left(n^{1/2}\right)$ (Banerjee et al. 2021)
Nash Social Welfare ($p = 0$)	$8\log^3(2n)$	$O(\log^3 n)$	$\Omega\left(\log^{1-\varepsilon} n\right)$ (Banerjee et al. 2021)
$p \in (-\infty, -1]$	$8\sqrt{n}\log(2n)$	$O(\sqrt{n}\log n)$	$\Omega\left(n^{\frac{\lvert p\rvert}{2\lvert p\rvert+1}}\right)$
$p \in (-1, -1/4]$	$8n^{\frac{\lvert p\rvert}{\lvert p\rvert+1}}\log^2(2n)$	$O\left(n^{\frac{\lvert p\rvert}{\lvert p\rvert+1}}\log^2 n\right)$	$\Omega\left(n^{\frac{\lvert p\rvert}{2\lvert p\rvert+1}}\right)$
$p \in (-1/4, \frac{-1}{\log(2n)}]$	$8(2n)^{2\lvert p\rvert}\log^3(2n)$	$O\left(n^{2\lvert p\rvert}\log^3 n\right)$	$2^{-(2+2/\lvert p\rvert)} \cdot n^{\frac{\lvert p\rvert}{2\lvert p\rvert+1}}$
$p \in (\frac{-1}{\log(2n)}, 0]$	$32\log^3(2n)$	$O(\log^3 n)$	> 1
$p \in (0, 1)$	$16\log^3(2n)$	$O(\log^3 n)$	> 1

Table 1: Upper and lower bounds on the competitive ratio for p-mean welfare maximization. Here, the lower bounds hold for any constant $\varepsilon > 0$.

the algorithmic template remains the same, the proofs for different ranges of p require distinct arguments. Due to space constraints, the proofs of the upper bounds mentioned in Table 1 are deferred to the full version of the paper (Barman, Khan, and Maiti 2021). Also, note that, for multiple ranges of the exponent parameter, the lower bounds in the table closely match the upper bounds.

Lower Bounds. The following two theorems (proved in the full version of the paper (Barman, Khan, and Maiti 2021)) provide the lower bounds mentioned in Table 1. Theorem 2 shows that sub-optimality is unavoidable for $p < 1$.[2] Theorem 3 provides a stronger negative result for all $p < 0$.

Theorem 2. *For any $p < 1$, the p-mean welfare maximization problem does not admit an online algorithm that computes optimal allocations, i.e., the competitive ratio of any online algorithm is strictly greater than one.*

Theorem 3. *For any $p < 0$, there does not exist an online algorithm with competitive ratio strictly less than $2^{-(2+2/\lvert p\rvert)} \cdot n^{\frac{\lvert p\rvert}{2\lvert p\rvert+1}}$ for the p-mean welfare maximization problem.*

2 Notation and Preliminaries

We study the problem of allocating T divisible goods among n agents in an online manner. Let $[n] := \{1, \ldots, n\}$ denote the set of agents and $[T] := \{1, \ldots, T\}$ denote the set of goods.

The divisible goods arrive online, one by one, in T rounds overall. The value of good $t \in T$, for every agent $i \in [n]$, is revealed only when the good arrives (i.e., in round t); specifically, let $v_i^t \in \mathbb{R}_{\geq 0}$ be the (nonnegative) value of good $t \in [T]$ for agent $i \in [n]$.

For each good t, the online algorithm must make an irrevocable allocation decision, i.e., assign the good fractionally among the agents. Let $x_i^t \in [0,1]$ denote the fraction of the good $t \in [T]$ assigned to agent $i \in [n]$. Note that at most one unit of any good t is assigned among the agents: $\sum_{i=1}^n x_i^t \leq 1$. Furthermore, for each agent $i \in [n]$, a *bundle* $x_i := (x_i^t)_{t\in[T]} \in [0,1]^T$ refers to a tuple that denotes the fractional assignments of all the T goods to agent i.

An *allocation* $\mathbf{x} = (x_i)_{i\in[n]} \in [0,1]^{n\times T}$ refers to a fractional assignment of the goods among all the agents such that no more than one unit of any good is assigned.[3]

Throughout, we will assume that all the agents have *additive valuations*; in particular, for bundle $x_i = (x_i^t)_{t\in[T]} \in [0,1]^T$, agent i's valuation $v_i(x_i) := \sum_{t=1}^T x_i^t v_i^t$.

In this work, the extent of fairness and economic efficiency of allocations is measured by the *generalized means* of the values that the allocations generate among the agents. Specifically, with exponent parameter parameter $p \in (-\infty, 1]$, the pth generalized mean of n nonnegative numbers $\nu_1, \nu_2, \ldots, \nu_n \in \mathbb{R}_{\geq 0}$ is defined as

$$\mathrm{M}_p(\nu_1, \ldots, \nu_n) := \left(\frac{1}{n}\sum_{i=1}^n \nu_i^p\right)^{\frac{1}{p}}.$$

The generalized means $\mathrm{M}_p(\cdot)$ with $p \leq 1$, constitute a family of functions that capture multiple fairness and efficiency measures: $\mathrm{M}_p(\cdot)$ corresponds to the arithmetic mean when $p = 1$, and as p tends to zero, M_p—in the limit—is equal to the geometric mean. Also, $\lim_{p\to-\infty}\mathrm{M}_p(\nu_1, \ldots, \nu_n) = \min\{\nu_1, \ldots, \nu_n\}$. Therefore, following standard convention, we will write $\mathrm{M}_0(\nu_1, \ldots, \nu_n) := \left(\prod_{i=1}^n \nu_i\right)^{1/n}$ and $\mathrm{M}_{-\infty}(\nu_1, \ldots, \nu_n) := \min\{\nu_1, \ldots, \nu_n\}$.

Considering generalized means as a parameterized family of welfare objectives, we define the *p-mean welfare* $\mathrm{M}_p(\mathbf{x})$ of an allocation $\mathbf{x} = (x_i)_{i\in[n]} \in [0,1]^{n\times T}$ as

$$\mathrm{M}_p(\mathbf{x}) := \mathrm{M}_p\left(v_1(x_1), v_2(x_2), \ldots, v_n(x_n)\right) = \left(\frac{1}{n}\sum_{i=1}^n v_i(x_i)^p\right)^{1/p}$$

[2]By contrast, for $p = 1$, a greedy online algorithm (that assigns each good to the agent that values it the most) finds an allocation with maximum possible (average) social welfare.

[3]An online algorithm has an allocation (of all T goods) in hand only after the completion of all the T rounds.

Here, $\mathrm{M}_1(\mathbf{x})$ denotes the *average social welfare* of allocation $\mathbf{x}$ and $\mathrm{M}_0(\mathbf{x})$ denotes the allocation's *Nash social welfare*, $\mathrm{M}_0(\mathbf{x}) = \left(\prod_{i=1}^{n} v_i(x_i)\right)^{1/n}$. In addition, $\mathrm{M}_{-\infty}(\mathbf{x})$ denotes the *egalitarian welfare*, $\mathrm{M}_{-\infty}(\mathbf{x}) = \min_{i\in[n]} v_i(x_i)$.

We will assume throughout that for every agent the valuation of the grand bundle of goods $[T]$ is equal to one, i.e., for each $i \in [n]$, we have $\sum_{t=1}^{T} v_i^t = 1$. See Section 1 for a discussion on this scaling assumption. Also, without loss of generality, we will assume that, for all agents $i \in [n]$ and goods $t \in [T]$, the value $v_i^t \leq \frac{1}{n^2}$. This condition can be achieved online by considering—for each good—n^2 identical copies of value $1/n^2$ times the value of the underlying good. Note that our results hold in the adversarial model wherein the value of a good t can be set by an adaptive adversary (subject to the mentioned scaling) based upon the fractional assignments till round $(t-1)$.

Also, to ensure that certain numeric inequalities hold, we will throughout assume that $n \geq 16$.[4]

3 Online Algorithm

This section details our online algorithm, $\mathrm{ALG}(\Phi)$, for maximizing p-mean welfare, for $p \leq 1$. The algorithm operates with a given threshold Φ and upon the arrival of each good t, it distributes half of the good uniformly among the agents, i.e., the algorithm initializes fractional assignment $x_i^t = \frac{1}{2} \cdot \frac{1}{n}$, for each agent $i \in [n]$; see Line 3 in the algorithm and note that this initialization satisfies $\sum_{i=1}^{n} x_i^t = 1/2$. The remaining half of the good is further divided into $\log(2n)$ equal parts; in particular, for each $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$, the algorithm sets fractional assignments $x_i^{t,\alpha} \in [0,1]$ across the agents such that $\sum_{i=1}^{n} x_i^{t,\alpha} = \frac{1}{2} \cdot \frac{1}{\log(2n)}$. Note that such fractional assignments ensure that overall one unit of the good t is assigned across the agents.

For each $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$, to distribute $\frac{1}{2\log(2n)}$ fraction of the good t, the algorithm maintains two subsets of agents: A_t^α, referred to as the active set of agents, and B_t^α, a "vulnerable" subset of agents. For each α and good t, an agent i is said to be active and, hence, included in A_t^α, iff so far from the α part agent i has received value less than α/Φ, i.e., iff $\sum_{s=1}^{t-1} v_i^s x_i^{s,\alpha} < \alpha/\Phi$. For any α, only active agents continue to receive nonzero fractional assignments. Specifically, for the active agent a that values the current good t the most (see Line 6 in the algorithm), we set $x_a^{t,\alpha} = \frac{1}{2} \cdot \frac{1}{2\log(2n)}$. Furthermore, the algorithm considers a subset of the active agents $B_t^\alpha \subseteq A_t^\alpha$ for whom the pending goods cumulatively have value less than $\alpha/4$; see Line 7 in the algorithm and recall the scaling that $\sum_{s=1}^{T} v_\ell^s = 1$ for all agents ℓ. The remaining $\left(\frac{1}{4\log(2n)}\right)$ part of the good is uniformly distributed among the agents in B_t^α. At a high level, the algorithm maintains a set of active agents (who have yet to receive a sufficiently high value) and a subset of vulnerable agents (for whom limited value is left among the pending goods). The algorithm then strikes a balance between greedily assigning the good (Line 6) and uniformly distributing it among the vulnerable agents (Line 7). Indeed, the algorithm is computationally efficient and conceptually simple–we consider this as a relevant contribution, since such features lend the algorithm to large-scale implementations and explainable adaptations.

Algorithm 1: $\mathrm{ALG}(\Phi)$

1: Initialize index $t = 1$ and, for each (dyadic) $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$, initialize sets $A_t^\alpha = [n]$ and $B_t^\alpha = \emptyset$.
2: **for** each good $t = 1$ to T **do**
3:     Initialize fractional assignment $x_i^t = \frac{1}{2n}$, for each agent $i \in [n]$.
4:     **for** all $k = 1$ to $\log(2n)$ **do**
5:         Set $\alpha = \frac{1}{2^k}$ and initialize fractional assignment $x_i^{t,\alpha} = 0$, for each agent $i \in [n]$.
6:         Select agent $a = \arg\max_{j \in A_t^\alpha} v_j^t$ and assign $x_a^{t,\alpha} = \frac{1}{4\log(2n)}$.
7:         For each agent $i \in B_t^\alpha$, update $x_i^{t,\alpha} \leftarrow x_i^{t,\alpha} + \frac{1}{4\log(2n)} \frac{1}{|B_t^\alpha|}$.
8:         Set $A_{t+1}^\alpha = A_t^\alpha \setminus \left\{ j \in [n] : \sum_{s=1}^{t} v_j^s x_j^{s,\alpha} \geq \frac{\alpha}{\Phi} \right\}$.
9:         Set $B_{t+1}^\alpha = \left\{ \ell \in A_{t+1}^\alpha : \sum_{s=1}^{t} v_\ell^s > 1 - \frac{\alpha}{4} \right\}$.
10:     **end for**
11:     Update $x_i^t \leftarrow x_i^t + \sum_\alpha x_i^{t,\alpha}$ for all agents $i \in [n]$.
12: **end for**
13: **return** allocation $\mathbf{x} = (x_i^t)_{i,t}$.

Write $\mathbf{x} = (x_i)_{i\in[n]} \in [0,1]^{n\times T}$ to denote the allocation returned by $\mathrm{ALG}(\Phi)$; here, $x_i = (x_i^t)_{t\in[T]} \in [0,1]^T$ is the bundle assigned to agent $i \in [n]$.

Also, let $\boldsymbol{\omega} = (\omega_i)_{i\in[n]}$ be an arbitrary allocation wherein each agent $i \in [n]$ receives a bundle of value at least $1/2n$ but less than 1, i.e., $\frac{1}{2n} \leq v_i(\omega_i) < 1$. Next, we establish lemmas that hold for any such allocation $\boldsymbol{\omega}$ and any threshold $\Phi \leq n/4$. We will instantiate these lemmas with judicious choices of threshold Φ and for different values of the exponent parameter p; in each (p-specific) invocation, we will consider $\boldsymbol{\omega}$ as an allocation that approximately maximizes the p-mean welfare.

Remark 1. *Note that for each p, there exists an allocation $\boldsymbol{\omega} = (\omega_i)_i$ such that $\frac{1}{2n} \leq v_i(\omega_i) < 1$ and the p-mean welfare of $\boldsymbol{\omega}$ is at least half of the optimal p-mean welfare:*[5] *fix any $p \leq 1$, write $\widehat{\boldsymbol{\omega}} = (\widehat{\omega}_i^t)_{i,t}$ to denote an allocation that maximizes the p-mean welfare, and set $\omega_i^t = \frac{1}{2n} + \frac{1}{2}\widehat{\omega}_i^t$, for all i and t.*

At a high level, we aim to bound the number of agents that—in allocation $\mathbf{x}$—achieve value smaller than what they

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[4] Note that one can obtain a competitive ratio of n for p-mean welfare maximization by dividing every good uniformly among the n agents. Hence, for $n < 16$, a constant-factor competitive guarantee directly holds.

[5] Here, the upper bound $v_i(\omega_i) < 1$ follows from the scaling assumption.

achieve in $\boldsymbol{\omega}$. Upper bounding the number of such sub-optimal agents will enable us to establish p-mean welfare guarantees in subsequent sections.

For each $\alpha \in \left\{\frac{1}{2^k} : k \in [\log(2n)]\right\}$, let $H(\alpha, \boldsymbol{\omega})$ denote the subset of agents that achieve value at least α in allocation $\boldsymbol{\omega}$, i.e., $H(\alpha, \boldsymbol{\omega}) := \{i \in [n] : v_i(\omega_i) \geq \alpha\}$. Lemma 1 (below) shows that, at any point of time, at most $\frac{8n\log(2n)}{\Phi}$ of such high-valued agents are included in the set B_t^α (populated in Line 9 of ALG(Φ)).

Complementary to the set $H(\alpha, \boldsymbol{\omega})$, we define the set of agents $L(\alpha, \boldsymbol{\omega}) := \{i \in [n] : v_i(\omega_i) < \alpha\}$. Also, let $\widehat{L}(\alpha, \mathbf{x})$ denote the set of agents that achieve value less than $\frac{\alpha}{8\Phi}$ in $\mathbf{x}$, i.e., $\widehat{L}(\alpha, \mathbf{x}) := \left\{i \in [n] : v_i(x_i) < \frac{\alpha}{8\Phi}\right\}$. Lemma 2 (below) relates the number of such low-valued agents in the respective allocations.

In addition, comparing the values that agents receive in these two allocations, we will consider the subset of agents, $S_\Phi(\boldsymbol{\omega})$, that are (2Φ)-sub-optimal in $\mathbf{x}$; write $S_\Phi(\boldsymbol{\omega}) := \left\{i \in [n] : v_i(x_i) < \frac{1}{2\Phi} v_i(\omega_i)\right\}$.

Lemma 1. *For any iteration $t \leq T$ of* ALG(Φ) *and any $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$, we have*

$$|B_t^\alpha \cap H(\alpha, \boldsymbol{\omega})| \leq \frac{8n\log(2n)}{\Phi}.$$

Proof. Fix any iteration (good) t and $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$. Note that, if agent $i \in B_t^\alpha$, then $\sum_{s=t+1}^T v_i^s \leq \alpha/4$; see Line 9 in ALG($\Phi$) and recall that the valuations are scaled to satisfy $\sum_{t'=1}^T v_i^{t'} = 1$. Furthermore, for each agent $i \in H(\alpha, \boldsymbol{\omega})$, by definition, the value $v_i(\omega_i) = \sum_{s=1}^T \omega_i^s v_i^s \geq \alpha$. These inequalities imply that, for each agent $i \in H(\alpha, \boldsymbol{\omega}) \cap B_t^\alpha$, we have

$$\sum_{s=1}^t \omega_i^s v_i^s \geq \frac{3\alpha}{4} \tag{1}$$

Given that agent i is active during iteration t (specifically, $i \in B_t^\alpha \subseteq A_t^\alpha$), agent i must have been active ($i \in A_s^\alpha$) during all previous iterations $s \leq t$. Next, write a_s^α to denote the agent that was selected among active agents in iteration s (see Line 6 in ALG(Φ)), $a_s^\alpha := \arg\max_{j \in A_s^\alpha} v_j^s$. The fact that agent $i \in A_s^\alpha$ gives us $v_i^s \leq v_{a_s^\alpha}^s$ for each $s \leq t$. Using this bound and equation (1) we get

$$
\begin{aligned}
\frac{3\alpha}{4}\,|B_t^\alpha \cap H(\alpha, \boldsymbol{\omega})| &\leq \sum_{i \in B_t^\alpha \cap H(\alpha,\boldsymbol{\omega})} \left(\sum_{s=1}^t \omega_i^s v_i^s\right) && \text{(via (1))}\\
&= \sum_{s=1}^t \sum_{i \in B_t^\alpha \cap H(\alpha,\boldsymbol{\omega})} \omega_i^s\, v_i^s\\
&\leq \sum_{s=1}^t \sum_{i \in B_t^\alpha \cap H(\alpha,\boldsymbol{\omega})} \omega_i^s\, v_{a_s^\alpha}^s && \text{(since } v_i^s \leq v_{a_s^\alpha}^s)\\
&\leq \sum_{s=1}^t v_{a_s^\alpha}^s && \text{(since } \textstyle\sum_i \omega_i^s \leq 1 \text{ for each } s)\\
&\leq \sum_{s=1}^t 4\log(2n)\, x_{a_s^\alpha}^{s,\alpha}\, v_{a_s^\alpha}^s && \text{(since } x_{a_s^\alpha}^{s,\alpha} \geq \tfrac{1}{4\log(2n)}\text{; Line 6 in ALG}(\Phi))\\
&= 4\log(2n) \sum_{s=1}^t x_{a_s^\alpha}^{s,\alpha}\, v_{a_s^\alpha}^s\\
&\leq 4\log(2n) \sum_{s=1}^t \left(\sum_{j=1}^n x_j^{s,\alpha}\, v_j^s\right)\\
&= 4\log(2n) \sum_{j=1}^n \sum_{s=1}^t x_j^{s,\alpha}\, v_j^s
\end{aligned} \tag{2}
$$

We next bound the right-hand-side of the previous inequality by showing that $\sum_{s=1}^t x_j^{s,\alpha}\, v_j^s \leq \frac{3\alpha}{2\Phi}$, for all agents $j \in [n]$. Note that if $j \in A_t^\alpha$, then $\sum_{s=1}^t x_j^{s,\alpha}\, v_j^s \leq \frac{\alpha}{\Phi}$. Otherwise, if $j \in [n] \setminus A_t^\alpha$, then j was removed from the active set A_r^α during some iteration $r \leq t$ and we have $\sum_{s=1}^t x_j^{s,\alpha}\, v_j^s \leq \sum_{s=1}^{r-1} x_j^{s,\alpha}\, v_j^s + v_j^r \leq \frac{\alpha}{\Phi} + \frac{1}{n^2}$; the last inequality follows from the fact that the goods have value at most $1/n^2$. Since $\Phi \leq n/4$ and $\alpha \geq \frac{1}{2n}$, we get, for all agents $j \in [n]$:

$$\sum_{s=1}^t x_j^{s,\alpha}\, v_j^s \leq \frac{\alpha}{\Phi} + \frac{1}{n^2} \leq \frac{3\alpha}{2\Phi} \tag{3}$$

Equations (2) and (3) give us $\frac{3\alpha}{4}\,|B_t^\alpha \cap H(\alpha, \boldsymbol{\omega})| \leq 4\log(2n)\ \sum_{j=1}^n \left(\frac{3\alpha}{2\Phi}\right) = \frac{6\alpha n\log(2n)}{\Phi}$. Simplifying we obtain the desired bound $|B_t^\alpha \cap H(\alpha, \boldsymbol{\omega})| \leq \frac{8n\log(2n)}{\Phi}$. □

Recall that $L(\alpha, \boldsymbol{\omega}) := \{i \in [n] : v_i(\omega_i) < \alpha\}$ and $\widehat{L}(\alpha, \mathbf{x}) := \left\{i \in [n] : v_i(x_i) < \frac{\alpha}{8\Phi}\right\}$.

Lemma 2. *For any $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$ we have $|\widehat{L}\,(\alpha, \mathbf{x})\,| \leq |L(\alpha, \boldsymbol{\omega})| + \frac{8n\log(2n)}{\Phi}$.*

Proof. We begin by showing that $\widehat{L}(\alpha, \mathbf{x}) \subseteq B_T^\alpha$: consider any agent $i \in \widehat{L}(\alpha, \mathbf{x})$; by definition of this set, $v_i(x_i) < \frac{\alpha}{8\Phi}$ and, hence, agent $i \in A_T^\alpha$. In addition, given that the value of the good in the last round is at most $\frac{1}{n^2} \leq \frac{\alpha}{4}$, we get $i \in B_T^\alpha$. Since this containment holds for each agent $i \in \widehat{L}(\alpha, \mathbf{x})$, we obtain $\widehat{L}(\alpha, \mathbf{x}) \subseteq B_T^\alpha$.

This containment and Lemma 1 lead to the stated bound:

$$
\begin{aligned}
|\widehat{L}(\alpha, \mathbf{x})| &= |\widehat{L}(\alpha, \mathbf{x}) \cap L(\alpha, \boldsymbol{\omega})| + |\widehat{L}(\alpha, \mathbf{x}) \cap H(\alpha, \boldsymbol{\omega})|\\
& && (H(\alpha, \boldsymbol{\omega}) \text{ and } L(\alpha, \boldsymbol{\omega}) \text{ partition } [n])\\
&\leq |L(\alpha, \boldsymbol{\omega})| + |\widehat{L}(\alpha, \mathbf{x}) \cap H(\alpha, \boldsymbol{\omega})|\\
&\leq |L(\alpha, \boldsymbol{\omega})| + |B_T^\alpha \cap H(\alpha, \boldsymbol{\omega})| && \text{(since } \widehat{L}(\alpha, \mathbf{x}) \subseteq B_T^\alpha)\\
&\leq |L(\alpha, \boldsymbol{\omega})| + \frac{8n\log(2n)}{\Phi} && \text{(Lemma 1)}
\end{aligned}
$$

This completes the proof. □

Recall that $S_\Phi(\boldsymbol{\omega}) := \left\{i \in [n] : v_i(x_i) < \frac{1}{2\Phi} v_i(\omega_i)\right\}$. The following lemma establishes that the number of such sub-optimal agents decreases linearly with the chosen parameter Φ.

Lemma 3. *The number of sub-optimal agents* $|S_\Phi(\boldsymbol{\omega})| \leq \frac{8n \log^2(2n)}{\Phi}$.

Proof. We partition the set of agents $S_\Phi(\boldsymbol{\omega})$ into $\log(2n)$ subsets, based on the values $v_i(\omega_i)$s. Specifically, for each $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$, write subset $S(\alpha) := \{i \in S_\Phi(\boldsymbol{\omega}) : \alpha \leq v_i(\omega_i) < 2\alpha\}$. Since $\frac{1}{2n} \leq v_j(\omega_j) < 1$, for all agents $j \in [n]$, the subsets $S(\alpha)$s form a partition of $S_\Phi(\boldsymbol{\omega})$. Therefore, $\sum_{k=1}^{\log(2n)} \left|S\left(\frac{1}{2^k}\right)\right| = |S_\Phi(\boldsymbol{\omega})|$.

Next, we will show that $S(\alpha) \subseteq B_T^\alpha \cap H(\alpha, \boldsymbol{\omega})$ and apply Lemma 1. Towards this, note that for each agent $i \in S(\alpha) \subseteq S_\Phi(\boldsymbol{\omega})$, we have $v_i(x_i) < \frac{v_i(\omega_i)}{2\Phi} < \frac{2\alpha}{2\Phi} = \frac{\alpha}{\Phi}$. Since $\sum_{s=1}^T x_i^{s,\alpha} v_i^s \leq v_i(x_i)$, we get that agent i continues to be in the active set (for α) throughout the execution of the algorithm, i.e., $i \in A_T^\alpha$. In fact, agent $i \in B_T^\alpha$, since the value of the last good T is at most $\frac{1}{n^2} \leq \frac{1}{8n}$, which in turn in upper bounded by $\frac{\alpha}{4}$ (see Line 9 in $\text{ALG}(\Phi)$) since $\alpha \geq \frac{1}{2n}$; recall the assumption that $n \geq 16$.

Furthermore, for each agent $i \in S(\alpha)$, we have $v_i(\omega_i) \geq \alpha$, i.e., $i \in H(\alpha, \boldsymbol{\omega})$. These observations imply that every agent $i \in S(\alpha)$ is contained in $B_T^\alpha \cap H(\alpha, \boldsymbol{\omega})$; equivalently, $S(\alpha) \subseteq B_T^\alpha \cap H(\alpha, \boldsymbol{\omega})$. Therefore, Lemma 1 gives us $|S(\alpha)| \leq \frac{8n \log(2n)}{\Phi}$, for all $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$.

This establishes the Lemma: $|S_\Phi(\boldsymbol{\omega})| = \sum_{k=1}^{\log(2n)} \left|S\left(\frac{1}{2^k}\right)\right| \leq \log(2n) \frac{8n \log(2n)}{\Phi} = \frac{8n \log^2(2n)}{\Phi}$. ☐

4 Universal Online Algorithm for p-Mean Welfare

Considering the p-mean welfare maximization problem simultaneously for all $p \leq 1$, this section establishes an online algorithm with a universal competitive ratio of $O(\sqrt{n} \log n)$. Specifically, we establish Theorem 1 by executing our algorithm, $\text{ALG}(\Phi)$, with threshold $\Phi = 8\sqrt{n}\ \log(2n)$.

Throughout this section we will consider the allocation $\mathbf{x} = (x_i)_i$ returned by $\text{ALG}\left(8\sqrt{n} \log(2n)\right)$ and establish an $O(\sqrt{n} \log n)$-competitive ratio for different values of p: Subsection 4.1 details the guarantee for egalitarian welfare ($p = -\infty$) and Subsection 4.2 for Nash social welfare ($p = 0$). The proofs for $p \leq -1$ and $p > -1$ are deferred to the full version of the paper. Together, Subsections 4.1 and 4.2 along with the proofs for $p \leq -1$ and $p > -1$ in the full version prove Theorem 1.

Consider an arbitrary allocation $\boldsymbol{\omega} = (\omega_i)_i$ with the property that $\frac{1}{2n} \leq v_i(\omega_i) < 1$, for all agents $i \in [n]$, and recall that, complementary to $H(\alpha, \boldsymbol{\omega})$, the set $L(\alpha, \boldsymbol{\omega}) := \{i \in [n] : v_i(\omega_i) < \alpha\}$. Also, with $\Phi = 8\sqrt{n}\ \log(2n)$, the set $\widehat{L}(\alpha, \mathbf{x})$ contains the agents that have value less than $\frac{\alpha}{8\Phi} = \frac{\alpha}{64\sqrt{n}\log(2n)}$ in allocation $\mathbf{x}$, i.e., $\widehat{L}(\alpha, \mathbf{x}) = \left\{i \in [n] : v_i(x_i) < \frac{\alpha}{64\sqrt{n}\log(2n)}\right\}$. The following lemma builds upon Lemma 2 specifically for $\Phi = 8\sqrt{n}\log(2n)$.

Lemma 4. *For parameter $\Phi = 8\sqrt{n}\log(2n)$ and any $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$, if $|L(\alpha, \boldsymbol{\omega})| \leq \sqrt{n}$, then $\widehat{L}(\alpha, \mathbf{x}) = \emptyset$.*

Proof. Using the upper bound $|L(\alpha, \boldsymbol{\omega})| \leq \sqrt{n}$, we will show that no agent $i \in [n]$ is contained in $\widehat{L}(\alpha, \mathbf{x})$. Towards this, note that if, in $\text{ALG}(\Phi)$, agent i was removed directly from the active set A_t^α in some iteration t, then $v_i(x_i) \geq \frac{\alpha}{\Phi} \geq \frac{\alpha}{64\sqrt{n}\log(2n)}$ (Line 8). In such a case, $i \notin \widehat{L}(\alpha, \mathbf{x})$. Hence, for the rest of the proof we assume that $i \in A_t^\alpha$ for all iterations $1 \leq t \leq T$. Let f be the iteration in which agent i was included in B_f^α for the first time, i.e., $\sum_{s=f-1}^T v_i^s \geq \frac{\alpha}{4}$ and $\sum_{s=f}^T v_i^s < \frac{\alpha}{4}$ (Line 9). Since all the goods have value at most $1/n^2$ and $\alpha \geq \frac{1}{2n} \geq \frac{8}{n^2}$, we get

$$\sum_{s=f}^T v_i^s = \sum_{s=f-1}^T v_i^s - v_i^{f-1} \geq \frac{\alpha}{4} - \frac{1}{n^2} \geq \frac{\alpha}{8} \tag{4}$$

Next, note that Lemma 1 gives us $|B_s^\alpha \cap H(\alpha, \boldsymbol{\omega})| \leq \frac{8n\log(2n)}{\Phi} = \sqrt{n}$ for all $s \geq f$. Furthermore, we have $|L(\alpha, \boldsymbol{\omega})| \leq \sqrt{n}$. Therefore, for all $s \geq f$, we can upper bound B_s^α as follows

$$\begin{aligned} |B_s^\alpha| &= |B_s^\alpha \cap H(\alpha, \boldsymbol{\omega})| + |B_s^\alpha \cap L(\alpha, \boldsymbol{\omega})| \\ &\leq \sqrt{n} + \sqrt{n} = 2\sqrt{n} \end{aligned} \tag{5}$$

Since $i \in B_s^\alpha$ for all $s \geq f$, agent i receives at least $\frac{1}{4\log(2n)|B_s^\alpha|}$ fraction of the good in every iteration $s \geq f$ (Line 9). Therefore, the value achieved by agent i is at least

$$\begin{aligned} \sum_{s=f}^T \frac{1}{4\log(2n)|B_s^\alpha|}\ v_i^s &\underset{\text{(via (5))}}{\geq} \frac{1}{8\sqrt{n}\log(2n)} \sum_{s=f}^T v_i^s \\ &\underset{\text{(via (4))}}{\geq} \frac{1}{8\sqrt{n}\log(2n)}\ \frac{\alpha}{8} \\ &= \frac{\alpha}{64\sqrt{n}\log(2n)}. \end{aligned}$$

Therefore, even in the current case $i \notin \widehat{L}(\alpha, \mathbf{x})$. This, overall, establishes that no agent i is contained in $\widehat{L}(\alpha, \mathbf{x})$ (i.e., $\widehat{L}(\alpha, \mathbf{x}) = \emptyset$). The lemma stands proved. ☐

The following lemma multiplicatively bounds the number of low-valued agents in allocation $\mathbf{x}$ in terms of the number of low-valued agents in $\boldsymbol{\omega}$.

Lemma 5. *For parameter $\Phi = 8\sqrt{n}\log(2n)$ and any $\alpha \in \left\{\frac{1}{2^k} : 1 \leq k \leq \log(2n)\right\}$, we have $|\widehat{L}(\alpha, \mathbf{x})| \leq 2|L(\alpha, \boldsymbol{\omega})|$.*

Proof. Lemma 4 ensures that, if $L(\alpha, \boldsymbol{\omega}) \leq \sqrt{n}$, then $|\widehat{L}(\alpha, \mathbf{x})| = 0$. Hence, in such a case, the stated bound holds: $|\widehat{L}(\alpha, \mathbf{x})| \leq 2|L(\alpha, \boldsymbol{\omega})|$. The complementary case (i.e., $|L(\alpha, \boldsymbol{\omega})| > \sqrt{n}$) is addressed by Lemma 2:

$$\begin{aligned} |\widehat{L}(\alpha, \mathbf{x})| &\leq |L(\alpha, \boldsymbol{\omega})| + \frac{8n\log(2n)}{\Phi} \\ &= |L(\alpha, \boldsymbol{\omega})| + \sqrt{n} \quad \text{(here } \Phi = 8\sqrt{n} \cdot \log(2n)\text{)} \\ &\leq 2|L(\alpha, \boldsymbol{\omega})|. \end{aligned}$$

This completes the proof. ☐

4.1 Universal Guarantee for Egalitarian Welfare

In this subsection we show that $\textsc{Alg}(\Phi)$, with $\Phi = 8\sqrt{n} \cdot \log(2n)$, achieves a competitive ratio of $O\left(\sqrt{n} \cdot \log n\right)$ for maximizing egalitarian welfare $\mathrm{M}_{-\infty}(\cdot)$.[6] Here, let $\boldsymbol{\omega} = (\omega_i)_i$ denote an allocation with egalitarian welfare, $\mathrm{M}_{-\infty}(\boldsymbol{\omega})$, at least half of the optimal egalitarian welfare and $\frac{1}{2n} \leq v_i(\omega_i) < 1$, for all $i \in [n]$; such an allocation is guaranteed to exist (Remark 1).

We will show that the allocation $\mathbf{x} = (x_i)_i$—computed by $\textsc{Alg}\left(8\sqrt{n}\log(2n)\right)$—satisfies $\mathrm{M}_{-\infty}(\mathbf{x}) \geq \frac{1}{128\sqrt{n}\log(2n)}\mathrm{M}_{-\infty}(\boldsymbol{\omega})$ and, hence, obtain the stated competitive ratio for egalitarian welfare.

Write κ to denote the integer that satisfies $\frac{1}{2^\kappa} \leq \mathrm{M}_{-\infty}(\boldsymbol{\omega}) < \frac{2}{2^\kappa}$. Since $v_i(\omega_i) \geq 1/2n$, for all agents $i \in [n]$, we have $\mathrm{M}_{-\infty}(\boldsymbol{\omega}) = \min_i v_i(\omega_i) \geq 1/2n$. Hence, $\kappa \leq \log(2n)$. Setting $\tilde{\alpha} := 1/2^\kappa$, we invoke Lemma 4 and note that $L(\tilde{\alpha}, \boldsymbol{\omega}) = \emptyset$ and, hence, $\widehat{L}(\tilde{\alpha}, \mathbf{x}) = \emptyset$. Therefore, all agents $i \in [n]$ satisfy $v_i(x_i) \geq \frac{\tilde{\alpha}}{64\sqrt{n}\log(2n)} > \frac{\mathrm{M}_{-\infty}(\boldsymbol{\omega})}{128\sqrt{n}\log(2n)}$. Equivalently, $\mathrm{M}_{-\infty}(\mathbf{x}) \geq \frac{1}{128\sqrt{n}\log(2n)}\mathrm{M}_{-\infty}(\boldsymbol{\omega})$ and the stated competitive ratio holds.

The next subsection shows that the allocation $\mathbf{x}$ obtains an analogous competitive guarantee for Nash social welfare as well.

4.2 Universal Guarantee for Nash Social Welfare

In this subsection, let $\boldsymbol{\omega} = (\omega_i)_i$ denote an allocation with Nash social welfare, $\mathrm{M}_0(\boldsymbol{\omega})$, at least half of the optimal Nash social welfare and $\frac{1}{2n} \leq v_i(\omega_i) < 1$, for all $i \in [n]$; such an allocation is guaranteed to exist (Remark 1). Recall that $S_\Phi(\boldsymbol{\omega})$ is a subset of agents that are (2Φ)-sub-optimal in $\mathbf{x}$; specifically, $S_\Phi(\boldsymbol{\omega}) := \left\{i \in [n] : v_i(x_i) < \frac{1}{2\Phi}v_i(\omega_i)\right\}$. For notational convenience, throughout this subsection we will use S for the set $S_\Phi(\boldsymbol{\omega})$ and write $S^c := [n] \setminus S$. In the current context we have $\Phi = 8\sqrt{n} \cdot \log(2n)$ and, hence, Lemma 3, gives us $|S| \leq \frac{8n\log^2(2n)}{\Phi} = \sqrt{n} \cdot \log(2n)$. Using this upper bound on $|S|$ we establish the competitive ratio for Nash social welfare:

$$
\begin{aligned}
\left(\frac{\prod_{i=1}^n v_i(\omega_i)}{\prod_{i=1}^n v_i(x_i)}\right)^{\frac{1}{n}} &= \left(\prod_{i=1}^n \frac{v_i(\omega_i)}{v_i(x_i)}\right)^{\frac{1}{n}} \\
&= \left(\prod_{i \in S} \frac{v_i(\omega_i)}{v_i(x_i)}\right)^{\frac{1}{n}} \left(\prod_{i \in S^c} \frac{v_i(\omega_i)}{v_i(x_i)}\right)^{\frac{1}{n}} \\
&\leq (2n)^{\frac{|S|}{n}} \left(\prod_{S^c} \frac{v_i(\omega_i)}{v_i(x_i)}\right)^{\frac{1}{n}} \\
&\qquad \left(\text{since } v_i(\omega_i) \leq 1 \ \&\ v_i(x_i) \geq \frac{1}{2n}, \text{ for all } i; \text{ Line 3}\right) \\
&\leq (2n)^{\frac{|S|}{n}} \left(16\sqrt{n}\log(2n)\right)^{\frac{S^c}{n}} \\
&\qquad \left(\text{since } v_i(x_i) \geq \frac{v_i(\omega_i)}{16\sqrt{n}\log(2n)} \text{ for all } i \in S^c\right) \\
&\leq (2n)^{\frac{\sqrt{n}\cdot\log(2n)}{n}} \left(16\sqrt{n}\log(2n)\right) \\
&\qquad \left(\text{since } |S| \leq \sqrt{n}\log(2n)\right) \\
&\leq 1328\sqrt{n}\log(2n) \\
&\qquad \left(\text{since } (2n)^{\frac{\sqrt{n}\cdot\log(2n)}{n}} \leq 83 \text{ for all } n \geq 1\right)
\end{aligned}
$$

Hence, the allocation $\mathbf{x}$ is $O\left(\sqrt{n}\log n\right)$-competitive for Nash Social Welfare.

Remark 2. *As mentioned previously, the universal guarantees for the ranges $p \leq -1$ and $p > 1$ appear in the full version of the paper. In addition, the full version contains the derivations of the the p-specific upper bounds (mentioned in Table 1). We note that, while we have a single algorithmic template, the derivations are distinct across different ranges of p.*

5 Conclusion and Future Work

This work studies online allocation of divisible goods and develops encompassing guarantees for p-mean welfare objectives. Our results hold under a standard (in the fair division literature) scaling assumption. Relaxing this assumption by, say, considering the problem in the algorithms-with-prediction framework (Mitzenmacher and Vassilvitskii 2021) is an interesting direction for future work. Another relevant direction would be to study online p-mean welfare maximization with stochastic valuations or in the random-order-arrival model. Connecting approximation guarantees for p-mean welfare and other well-studied fairness criteria, such as (bounded) envy, is a meaningful thread as well.

The current paper focussed on divisible goods. However, some of our results extend to settings wherein the goods cannot be fractionally assigned, i.e., extend to indivisible goods. In particular, under assumption that all the (indivisible) goods have sufficiently small values, one can obtain high-probability bounds for egalitarian welfare. Working with such (beyond worst case) assumptions and studying online p-mean welfare maximization for indivisible goods will also be interesting.

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[6]As mentioned previously, this competitive ratio is tight, up to a log factor.

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