Hedonic Diversity Games: A Complexity Picture with More than Two Colors

Robert Ganian[1], **Thekla Hamm**[1], **Dušan Knop**[2], **Šimon Schierreich**[2], **Ondřej Suchý**[2]
[1] Algorithms and Complexity Group, TU Wien, Austria
[2] Faculty of Information Technology, Czech Technical University in Prague, Czech Republic
{rganian, thamm}@ac.wien.ac.at, {dusan.knop, schiesim, ondrej.suchy}@fit.cvut.cz

Abstract

Hedonic diversity games are a variant of the classical Hedonic games designed to better model a variety of questions concerning diversity and fairness. Previous works mainly targeted the case with two diversity classes (represented as colors in the model) and provided some initial complexity-theoretic and existential results concerning Nash and individually stable outcomes. Here, we design new algorithms accompanied with lower bounds which provide a complete parameterized-complexity picture for computing Nash and individually stable outcomes with respect to the most natural parameterizations of the problem. Crucially, our results hold for general Hedonic diversity games where the number of colors is not necessarily restricted to two, and show that—apart from two trivial cases—a necessary condition for tractability in this setting is that the number of colors is bounded by the parameter. Moreover, for the special case of two colors we resolve an open question posed in previous work.

Introduction

Hedonic games are the prototypical example of coalition-forming games, which are games where agents seek to form coalitions in a way which satisfies their individual preferences (Drèze and Greenberg 1980; Elkind, Fanelli, and Flammini 2020; Kerkmann et al. 2020; Brandt, Bullinger, and Wilczynski 2021; Banerjee, Konishi, and Sönmez 2001). In classical Hedonic games, agents' preferences over possible coalitions take into account the specific agents present in the hypothetical coalition. Anonymous (Hedonic) games (Bogomolnaia and Jackson 2002) are a well-studied variant of the coalition-forming game where preferences only take into account the sizes of the coalitions, not their members.

Bredereck, Elkind, and Igarashi (2019) recently initiated the study of a model that generalizes and conceptually combines properties of Hedonic and anonymous games. In particular, they considered the scenario where each agent is assigned a class or *color*, and preference profiles are defined with respect to the ratios of classes occurring in the coalition. Loosely following their terminology, we call these *Hedonic diversity games*[1]. The question targeted by Hedonic diversity games occurs in a number of distinct settings, such as the Bakers and Millers game (where coalitions are formed between two types of agents with competing preferences) (Aziz et al. 2019; Schelling 1971; Bilò et al. 2018) or when the task is to assign guests from several backgrounds to tables (Boehmer and Elkind 2020b; Igarashi, Sliwinski, and Zick 2019).

Naturally, the most prominent computational question arising from the study of Hedonic diversity games targets the computation of an outcome that is stable under some well-defined notion of stability. While several stability concepts have been considered in the literature including, e.g., *envy-freeness* and *core stability*, here we focus on the two arguably most prominent notions of stability that have been applied in the context of Hedonic games:

- *Nash stability*, which ensures that no agent prefers leaving their coalition to join a different one, and
- *individual stability*, where no agent prefers leaving their coalition to join another coalition whose members would all appreciate or be indifferent to such a change.

In their pioneering work, Boehmer and Elkind (2020a) have provided initial results on the computational complexity of computing stable outcomes for Hedonic diversity games (a problem we hereinafter refer to as HDG), with a particular focus on the case with 2 colors. They analyzed the problem not only from the viewpoint of classical complexity, but also with respect to the more refined *parameterized* paradigm[2]; among others, they showed that HDG is NP-hard even when restricted to instances with 5 colors, and for the case of 2 colors they obtained a polynomial time algorithm for individually stable HDG and an XP algorithm for Nash-stable HDG when parameterized by the size of the smaller color class (Boehmer and Elkind 2020a, Theorem 5.2). Still, our understanding of the computational aspects of HDG has up to now remained highly incomplete: no

[1]This term was originally used for the special case of two classes, and the presence of $k > 2$ classes was previously identified by adding "*k-tuple*" or "*k-*"(Boehmer and Elkind 2020a). Since here we consider k to be part of the input, we use *Hedonic diversity games* for the general model.

[2]A brief introduction to parameterized complexity is provided in the Preliminaries.

tractable fragments of the problem have been identified beyond the aforementioned XP algorithm targeting Nash stability for 2 colors and the polynomial-time algorithm for individual stability for 2 colors. In fact, even whether this known XP-algorithm for 2 colors can be improved to a fixed-parameter one has been explicitly stated as an open problem (Boehmer and Elkind 2020a, Section 3).

Contribution. We obtain new algorithms and lower bounds that paint a comprehensive picture of the complexity of HDG through the lens of parameterized complexity. In particular, our results provide a complete understanding of the exact boundaries between tractable and intractable cases for HDG for both notions of stability and with respect to the most fundamental parameters that can either be identified from the input or have been used as additional conditions defining which outcomes are accepted:

- The number of colors on the input, denoted by γ;
- The maximum size σ of a coalition in an acceptable outcome. This is motivated by application scenarios as well as several works on the stable roommates problem—a special case of Hedonic games—restricted to coalitions of fixed size (Huang 2007; Ng and Hirschberg 1991; Bredereck et al. 2020);
- Two possible bounds on the maximum number of coalitions in an acceptable outcome: $\rho_{\geq 1}$ bounds the total number of coalitions *including* agents who are alone, while $\rho_{\geq 2}$ bounds the total number of coalitions *excluding* agents who are alone. Both of these restrictions induce different complexity-theoretic behaviors and are meaningful in different contexts—for instance, the former when coalitions represent tables at a conference dinner, while the latter when coalitions represent optional sports activities.
- The number τ of *agent types* of the instance, i.e., the number of different preference lists that need to be considered. Agent types have been used as a parameter in many related works (Bredereck et al. 2020; Biró, Irving, and Schlotter 2011; Kavitha, Nasre, and Nimbhorkar 2014; Irving, Manlove, and Scott 2008) and is a more relaxed parameterization than the number of agents. In the context of HDG, the number of agents bounds the size of the instance and hence is not an interesting parameter.

The complexity picture for HDG is provided in Figure 1 and is based on a set of three non-trivial algorithms and four reductions. Our results show that apart from two trivial cases (restricting the size and number of coalitions), a necessary condition for tractability under the considered parametrizations is that the number of colors is bounded by the parameter—and this condition is also sufficient, at least as far as XP-tractability is concerned. Remarkably, in our general setting the problem retains the same complexity regardless of whether we aim for Nash or individual stability—this contrasts the previously studied subcase of $\gamma = 2$ (Boehmer and Elkind 2020a).

Apart from requiring non-trivial insight into the structure of a possible solution, the approaches used to solve the tractable fragments vary greatly from one another: the fixed-parameter algorithm w.r.t. $\gamma + \sigma$ combines branching with an ILP formulation, the XP-algorithm w.r.t. $\gamma + \tau$ relies on advanced dynamic programming and the XP-algorithm which parameterizes by $\gamma + \rho_{\geq 2}$ combines careful branching with a network flow subroutine. We complement these positive results with lower bounds which show that none of the parameters can be dropped in any of the algorithms. This is achieved by a set of W[1]-hardness and NP-hardness reductions, each utilizing different ideas and starting from diverse problems: INDEPENDENT SET, MULTIDIMENSIONAL SUBSET SUM, EXACT COVER BY 3-SETS and PARTITION.

As our final result, we resolve an open question of Boehmer and Elkind (2020a): when the number of colors (γ) is 2, is Nash-stable HDG fixed-parameter tractable when parameterized by the size of the smaller color class? Here, we provide a highly non-elementary reduction from a variant of the GROUP ACTIVITY SELECTION problem (Darmann et al. 2017; Eiben, Ganian, and Ordyniak 2018) which excludes fixed-parameter tractability.

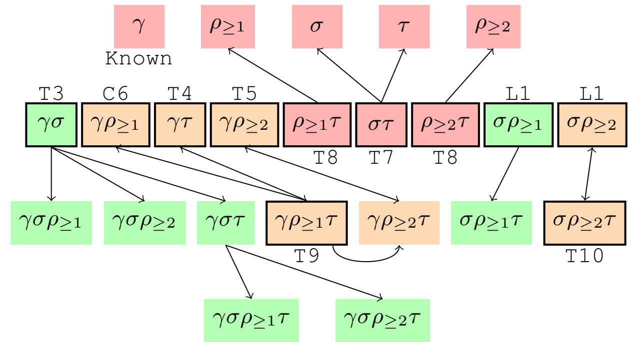

Figure 1: The complexity picture for HDG for both Nash and individual stability. Combinations of parameters which give rise to fixed-parameter algorithms are highlighted in green, while combinations for which HDG is W[1]-hard but in XP are highlighted in orange and NP-complete combinations are highlighted in red. Results explicitly proved in this work are represented by a black box and a reference to the given theorem, corollary or lemma.

Preliminaries

For integers $i < j$, we let $[i] = \{1, \ldots, i\}$, $[i]_0 = [i] \cup \{0\}$ and $[i, j] = \{i, \ldots, j\}$. $\mathbb{N}$ denotes the set of positive integers.

Hedonic Diversity Games. Let $N = [n]$ be a set of agents partitioned into color classes $D_1, \ldots, D_\gamma$. For a set $X \subseteq N$, we define the *palette* of X as the tuple $\left(\frac{|D_c \cap X|}{|X|}\right)_{c \in [\gamma]}$. Let M be a set of weak orders over the set of all palettes, and let $\succeq$ be a (not necessarily surjective) mapping which assigns each agent $i \in N$ to its preference list $\succeq_i \in M$.

A subset $C \subseteq N$ is called a *coalition*, and if $|C| = 1$ we call it a *trivial coalition*. Let $\Pi = \{C_1, \ldots, C_\ell\}$ be a partitioning of the agents, i.e., a set of subsets of N such that $\bigcup_{i=1}^{\ell} C_i = N$ and all C_i's are pairwise disjoint. We call Π an *outcome*, and use Π_i to denote the coalition the agent i is involved in for the outcome Π.

The notion of *stability* of an outcome Π for a game

$(N, (\succeq_i)_{i \in N})$ will be crucial for our considerations. If there is an agent i and coalition C in Π (possibly allowing $C = \emptyset$) such that $C \cup \{i\} \succ_i \Pi_i$, we say i admits a *NS-deviation* to C. Π is called *Nash stable* (NS) if it contains no agent with a NS-deviation. If agent i admits a NS-deviation to C where in addition for each agent $j \in C$ it holds that $C \cup \{i\} \succeq_j C$, we say that i admits an *IS-deviation* to C. Π is then called *individually stable* (IS) if it contains no agent with an IS-deviation. The core computational task that arises in the study of Hedonic diversity games is determining whether an instance admits a stable outcome w.r.t. the chosen notion of stability.

The classical notion of Hedonic games coincides with the special case of Hedonic diversity games where each agent receives their own color. Since already for Hedonic games the preference profiles may be exponentially larger than N even after removing all coalitions that are strictly less favorable than being alone, and this blow-up does not reflect the usual application scenarios for Hedonic games, we adopt the *oracle model* that has been proposed in previous works (Peters 2016a,b; Igarashi and Elkind 2016): instead of having the preference profiles included on the input, we are provided with an oracle that can be queried to determine (in constant time) whether an agent $i \in N$ prefers some coalition to another coalition.

Our aim will be to obtain an understanding of the problem's complexity with respect to the following parameters and their combinations:

- γ the number of color classes,
- σ the maximum size of a coalition,
- $\rho_{\geq 1}$ the maximum number of coalitions,
- $\rho_{\geq 2}$ the maximum number of non-trivial coalitions,
- τ the number of *agent types*, formally defined as $|M|$.

We can now formalize our problems of interest:

HDG-NASH	
Input:	Instance $\mathcal{I}$ consisting of a set $N = [n]$ of agents partitioned into $D_1, \ldots, D_\gamma$, an oracle that can compare coalitions according to $\succeq_i$ for each $i \in N$, and integers σ, $\rho_{\geq 1}$, $\rho_{\geq 2}$.
Question:	Does $\mathcal{I}$ admit a Nash stable outcome consisting of at most $\rho_{\geq 1}$ coalitions and $\rho_{\geq 2}$ non-trivial coalitions, each of size at most σ?

HDG-INDIVIDUAL is then defined analogously, with the distinction that the outcome must be individually stable; we use HDG to jointly refer to both problems. Observe that the restrictions imposed by σ, $\rho_{\geq 1}$, $\rho_{\geq 2}$ can be made irrelevant by setting these integers to n, meaning that HDG-NASH and HDG-INDIVIDUAL generalize the case where these three restrictions are removed. In fact, as a byproduct of our results we obtain that whenever any of these integers is not considered a parameter but merely as part of the input, removing the restriction completely does not affect the problem's complexity.

Example 1. Let $\mathcal{I}$ be an instance with 4 agents, denoted a, b, c, d for clarity, such that $D_1 = \{a, b\}$ and $D_2 = \{c, d\}$. The agents have the following preference lists:

$$a :: (\tfrac{1}{3}, \tfrac{2}{3}) \succ (\tfrac{2}{3}, \tfrac{1}{3}) \succ (1, 0) \succ (\tfrac{1}{2}, \tfrac{1}{2})$$
$$b :: (\tfrac{1}{2}, \tfrac{1}{2}) \succ (\tfrac{1}{3}, \tfrac{2}{3}) \succ (1, 0) \succ (\tfrac{2}{3}, \tfrac{1}{3})$$
$$c, d :: (\tfrac{1}{3}, \tfrac{2}{3}) \succ (\tfrac{2}{3}, \tfrac{1}{3}) \succ (\tfrac{1}{2}, \tfrac{1}{2}) \succ (0, 1)$$

One can see that the preferences of agents a, c, and d are derivable from the same master list

$$(\tfrac{1}{3}, \tfrac{2}{3}) \succ (\tfrac{2}{3}, \tfrac{1}{3}) \succ (1, 0) \succ (\tfrac{1}{2}, \tfrac{1}{2}) \succ (0, 1),$$

whereas preference list of agent b is not, i.e., $\mathcal{I}$ has exactly two types of agents and two colors. However, the color classes and type classes do not coincide.

If we are interested in an individually stable outcome, we can choose $C_1 = \{a, c, d\}$ and $C_2 = \{b\}$. Note that this outcome is not Nash stable, since b prefers to be in the grand coalition $\{a, b, c, d\}$ and, in contrast to IS-stability, it does not matter whether other agents are willing to accept b. On the other hand, the outcome $C_1 = \{b, c, d\}$, $C_2 = \{a\}$ is Nash stable (and hence also individually stable).

Parameterized Complexity. Parameterized complexity (Cygan et al. 2015; Downey and Fellows 2013; Niedermeier 2006) analyzes the running time of algorithms with respect to a parameter $k \in \mathbb{N}$ and input size n. The high-level idea is to find a parameter that describes the structure of the instance such that the combinatorial explosion can be confined to this parameter. In this respect, the most favorable complexity class is FPT (*fixed-parameter tractable*) which contains all problems that can be decided by an algorithm running in $f(k) \cdot n^{\mathcal{O}(1)}$ time, where f is a computable function. Algorithms with this running-time are called *fixed-parameter algorithms*. A less favorable outcome is an XP *algorithm*, which is an algorithm running in $n^{f(k)}$ time; problems admitting such algorithms belong to the class XP. Naturally, it may also happen that an NP-complete problem remains NP-hard even for a fixed value of k, in which case we call the problem *para*-NP-*complete*.

Showing that a problem is W[1]-hard rules out the existence of a fixed-parameter algorithm under the well-established assumption that W[1] $\neq$ FPT. This is done via a so-called *parameterized reduction* (Cygan et al. 2015).

Algorithms and Tractable Fragments

The aim of this section is to establish the algorithmic upper bounds that form the foundation for the complexity picture provided in Figure 1. We begin with a fairly simple observation that establishes our first two tractable fragments.

Lemma 1. HDG-NASH *and* HDG-INDIVIDUAL *are:*

1. *fixed-parameter tractable parameterized by* $\rho_{\geq 1} + \sigma$, *and*
2. *in* XP *parameterized by* $\rho_{\geq 2} + \sigma$.

Proof Sketch. For the first claim, note that either $n \leq \rho_{\geq 1} \cdot \sigma$—in which case we may brute-force over all possible outcomes in time at most $\mathcal{O}(\sigma \rho_{\geq 1}{}^{\sigma \rho_{\geq 1} + 2})$—or we are facing an obvious No-instance.

For the second claim, we begin by branching to determine the structure of non-trivial coalitions in a solution, that is, we branch on how many non-trivial coalitions there will be in the outcome and for each such coalition we branch to determine its size. Now in each branch we have at most $\rho_{\geq 2} \cdot \sigma$ "positions" for agents in the identified non-trivial coalitions, and we apply an additional round of branching to determine which agents will occupy these positions. At this point, the outcome is fully determined and we can check whether it is stable and satisfies the requirements. This results in an algorithm with a total running time of at most $n^{\mathcal{O}(\rho_{\geq 2} \cdot \sigma)}$. □

Our first non-trivial result is an algorithm for solving HDG parameterized by the number of colors and the coalition size. There, we first observe that in this case we may assume the number of types to also be bounded.

Observation 2. *It holds that* $\tau \leq (\gamma^{\sigma+1})! \cdot 2^{\gamma^{\sigma+1}}$.

Theorem 3. HDG-NASH *and* HDG-INDIVIDUAL *are fixed-parameter tractable parameterized by* $\gamma + \sigma$.

Proof Sketch. Let $\mathcal{C}$ be the set of all possible types of coalitions, where each type of coalition C is specified by the set of the combination of agent type and color of each agent in a coalition of type C. Observe that $|\mathcal{C}| \leq (\tau \cdot \gamma)^{\sigma+1}$, which by Observation 2 means that $|\mathcal{C}| \leq 2^{\gamma^{\mathcal{O}(\sigma)}}$. For each type of coalition $C \in \mathcal{C}$ we now branch to determine whether it occurs 0, 1, or at least 2 times in the solution, i.e., in a hypothetical Nash-stable or individually stable outcome. For each type of coalition C, we store the condition on the number of its occurrences imposed by this branching via the variable $\pi_C \in \{0, 1, 2^+\}$; an outcome *respects* the branch if each type of coalition occurs in Π π_C many times. Observe that the information given by the variables π_C is sufficient to determine whether an outcome respecting the branch will be (Nash or individually) stable. In fact:

Claim 1. *It is possible to verify, in time at most* $\sigma \cdot |\mathcal{C}|^2$*, whether every outcome respecting the branch* $(\pi_C)_{C\in\mathcal{C}}$ *is (Nash or individually) stable, or whether no such outcome is (Nash or individually) stable.*

At this point it remains to determine whether there exists an outcome Π respecting the branch $(\pi_C)_{C\in\mathcal{C}}$. We resolve this by encoding the question into an Integer Linear Program (ILP) with boundedly-many constraints.

In particular, the ILP contains one variable per $C \in \mathcal{C}$ specifying how many times each type of coalition occurs in the outcome. Constraints are used to ensure that the agents can be partitioned into coalition such that the requirements on the number of types of coalitions are satisfied. While the number of variables is large, the ILP has at most $\gamma \cdot \tau + 2$ constraints and the largest coefficient is upper-bounded by σ, allowing us to solve it using known algorithms in time at most $2^{2^{\gamma^{\mathcal{O}(\sigma)}}} \cdot n$ (Eisenbrand and Weismantel 2020; Jansen and Rohwedder 2019), which also upper-bounds the total runtime of the algorithm. □

Next, we turn our attention to HDG parameterized by the number of colors and agent types. While our previous result reduced the problem to a tractable fragment of ILP, here we will employ a non-trivial dynamic programming subroutine.

Theorem 4. HDG-NASH *and* HDG-INDIVIDUAL *are in* XP *parameterized by* $\gamma + \tau$.

Proof Sketch. For each color $c \in [\gamma]$ and agent type t (represented as an integer in $[\tau]$), let $n_{c,t}$ be the number of agents of color c and type t in the instance (the type of each agent can be determined using $\mathcal{O}(n^{2\gamma})$ oracle calls). It will be useful to observe that agents of the same type and color are pairwise interchangeable without affecting the stability of an outcome.

For each combination of $c \in [\gamma]$ and $t \in [\tau]$, we now branch to determine the "worst" and "second-worst" coalitions in which an agent of color c and type t appears in a sought-after NS/IS outcome. Formally, this branching identifies $(\gamma \cdot \tau)$-many coalitions $C_1^{c,t}$ and $C_2^{c,t}$, each represented as a palette $\left(\frac{|C_\star^{c,t} \cap D_1|}{|C_\star^{c,t}|}, \ldots, \frac{|C_\star^{c,t} \cap D_\gamma|}{|C_\star^{c,t}|}\right)$, such that $C_2^{c,t} \succeq_t C_1^{c,t}$. Since there are at most $(n+1)^\gamma$ many such palettes, the number of branches is upper-bounded by $\left((n+1)^{2\gamma^2 \cdot \tau}\right)$.

Our aim is to determine whether it is possible to pack agents into stable coalitions, whereas the insight we use is that the information we branched on allows us to predetermine whether a coalition will be stable or not. Intuitively, the coalitions should not be "worse" than the branch allows (only one coalition as "bad" as $C_1^{c,t}$ is allowed, unless $C_1^{c,t} \sim_t C_2^{c,t}$, and the others must be at least as "good" as $C_2^{c,t}$) ensuring that they do not become the source of a deviation and should not represent a destination for a deviation from coalitions as "bad" as $C_1^{c,t}$ or $C_2^{c,t}$. We solve this packing problem via dynamic programming.

We can now provide a high-level description of the dynamic programming procedure. The procedure constructs coalitions one by one, and stores information about the constructed coalitions in the form of "patterns" which capture all the required information to prevent undesirable deviations. Crucially, the number of patterns is bounded by $n^{\mathcal{O}(\gamma \cdot \tau)}$, and the program stores a table which keeps track of whether each pattern can be realized or not. As soon as the algorithm constructs a pattern which corresponds to a solution we output Yes, while if the algorithm reaches a stage where it cannot identify any new realized patterns we output No. The total running time can be upper-bounded by $n^{\mathcal{O}(\gamma^2 \cdot \tau)}$. □

The final result in this section is an XP-algorithm for HDG parameterized by the number of colors and non-trivial coalitions by combining branching with a flow subroutine.

Theorem 5. HDG-NASH *and* HDG-INDIVIDUAL *are in* XP *parameterized by* $\gamma + \rho_{\geq 2}$.

Proof Sketch. First, consider the case of Nash stability. We begin by branching to determine the number of non-trivial coalitions (at most $\rho_{\geq 2}$ many), and for each non-trivial coalition and each color we branch to determine the number of agents of that color in the coalition; this, in total, yields a branching factor of at most $\rho_{\geq 2} \cdot (n+1)^{\rho_{\geq 2} \cdot \gamma}$. We will view each branch as a "guess" of these properties of a hypothetical targeted Nash-stable outcome. We perform a set of simple consistency checks and discard guesses which do not pass these.

Our task is now to determine where each agent $a \in N$ can be placed. To this end, we build an auxiliary bipartite graph whose vertices are agents on the one side and coalition-color pairs on the other side. Let $\mathcal{C}$ be the set corresponding to our guessed coalitions (both trivial and non-trivial ones). We say that $C \in \mathcal{C}$ is *valid* for an agent $a \in N$ of color c iff

- c should appear in the coalition corresponding to C and
- a weakly prefers the coalition corresponding to C over a coalition arising from adding a to any coalition corresponding to some $\hat{C} \in \mathcal{C}$ or the empty coalition; (all the information required to determine this is the color composition of the coalition corresponding to C which we have already branched on).

The algorithm now simply verifies whether there is a way to assign each agent to a coalition-color pair using network flows, namely by connecting a source vertex to all agents and a sink vertex to all coalition-color pairs; edges are used to encode valid pairs and coalition-color pairs have capacities that correspond to the currently guessed number of agents of that color in the respective coalition.

This concludes the description of the algorithm for Nash stability. For individual stability, it is necessary to expand the initial branching in order to determine a small number of special "blocking" agents in each coalition; each blocking agent prevents all agents of a certain color from deviating to the coalition corresponding to C. This information is then taken into account when constructing the network flow instance in the second part of the algorithm. ☐

Observant readers will notice that there is one more tractability result in Figure 1, notably an XP-algorithm parameterized by $\gamma + \rho_{\geq 1}$. Since we may assume that, w.l.o.g., $\rho_{\geq 1} \geq \rho_{\geq 2}$, this follows as a corollary of Theorem 5.

Corollary 6. HDG-NASH *and* HDG-INDIVIDUAL *are in* XP *parameterized by* $\gamma + \rho_{\geq 1}$.

Algorithmic Lower Bounds

This section complements our algorithms and provides all the remaining results required to obtain a complexity picture for HDG. We begin by recalling that HDG remains NP-complete even when $\gamma = 2$ (Boehmer and Elkind 2020a). The first two results in this section establish the remaining para-NP-complete cases.

Before proceeding to the reductions, we build a general trap gadget that is similar in spirit to (Boehmer and Elkind 2020a, Example 5.3) and may be of independent interest. The gadget uses agents of three colors—red, blue, and green; we assume these colors are distinct from the colors used outside this gadget. Let R denote the red agent, B the blue agent, and G the green agents of which there are m in total; here G^+R is the profile $G^mR \succ \cdots \succ GGR \succ GR$, and similarly for G^+B and G^+. We will use $\mathcal{M}_G$ to denote all "desirable" coalitions containing at least one agent outside of the gadget, i.e., one agent that is neither red nor blue nor green.

$$R :: RB \succ G^+R \succ R \succ \cdots$$
$$B :: G^+B \succ RB \succ B \succ \cdots$$
$$\{G\} :: \mathcal{M}_G \succ G^+R \succ G^+B \succ G^+ \succ \cdots$$

Note that one can design this gadget using palettes if and only if one can design $\mathcal{M}_G$ in this way. Furthermore, observe that this can be done using 3 additional master lists.

Claim 2. *If a green agent is not part of a coalition in* $\mathcal{M}_G$*, then the outcome is neither IS nor NS.*

Theorem 7. HDG-NASH *and* HDG-INDIVIDUAL *are* NP-*hard even when* $\sigma = \tau = 4$.

Proof Sketch. We present a reduction from a classical NP-complete problem (Garey and Johnson 1979) called EXACT COVER BY 3-SETS (X3C): given a set $U = \{1, \ldots, 3m\}$ and a family $\mathcal{X} \subseteq \binom{U}{3}$, does there exist a set $S \subseteq \mathcal{X}$ such that every element of U occurs in exactly one element of S?

Let $\mathcal{I} = (U, \mathcal{X})$ be an instance of X3C with $|U| = 3m$. We introduce an agent a_u for every $u \in U$ with the (new) color u; we call these *universe agents*. We want these agents to stay in a coalition in $\mathcal{X}$ together with one green agent, i.e., in a coalitions of the following kind $\{X \cup \{G\} \mid X \in \mathcal{X}\}$, where G is a green agent. For these agents we present a single master list: $\{X \cup \{G\} \mid X \in \mathcal{X}\} \succ U$. That is, the agents can either be part of a desired coalition, or alone. We now apply the trap gadget construction described above to prevent stable outcomes where agents are alone—in particular, we will use $\mathcal{M}_G = \{X \cup \{G\} \mid X \in \mathcal{X}\}$.

To conclude the proof, it now suffices to verify that the constructed HDG is Nash-stable (and also individually stable) if and only if $\mathcal{I}$ was a Yes-instance. ☐

The proof for our second intractable case is based on a reduction from the NP-complete PARTITION problem (Garey and Johnson 1979): given a multiset $\mathcal{S}$ of integers, is there a partition of $\mathcal{S}$ into two distinct subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ such that the sum of elements in $\mathcal{S}_1$ equals the sum of numbers in $\mathcal{S}_2$?

Theorem 8. HDG-NASH *and* HDG-INDIVIDUAL *are* NP-*hard when restricted to instances where* τ*,* $\rho_{\geq 2}$ *and* $\rho_{\geq 1}$ *are upper-bounded by* 4.

Proof Sketch. For every instance of the PARTITION problem we construct an equivalent instance of the HDG as follows. For every number $i \in \mathcal{S}$ we introduce as many normal agents a_i as is the multiplicity of i in $\mathcal{S}$, and we assign every agent a_i color i. For the normal agents, we introduce a single trichotomous master list such that on the first (i.e., most preferred) place there are all coalitions where the sum of colors of its members equals to exactly k. On the second place, all the agents want to be alone. To complete the construction, we complement the HDG instance with a trap gadget that prevents agents from being alone. ☐

We remark that the instances produced by the reductions presented in the proofs of Theorems 7 and 8 never admit any stable outcome that violates the restrictions imposed by σ,

$\rho_{\geq 2}$ and $\rho_{\geq 1}$; in other words, the hardness also holds when we remove these restrictions.

At this point, we know that none of our XP algorithms can be strengthened by dropping a parameter in the parameterization. The remaining task for this section is to show the same for fixed-parameter algorithms, notably via matching W[1]-hardness reductions.

Our first lower bound excluding fixed-parameter tractability targets the parameterization by the number of colors, number of types, and the number of coalitions. To this end, we provide a reduction from a partitioned variant of a problem called MULTIDIMENSIONAL SUBSET SUM (MSS): given sets $S_1, \ldots, S_\omega$ of k-dimensional vectors of non-negative integers, a target k-dimensional vector t of non-negative integers, determine whether there exists a tuple $S' = (s_1, \ldots, s_\omega)$, $\forall i \in [\omega] : s_i \in S_i \cup \{\emptyset\}$ such that the vectors in S' sum up to t. While not stated explicitly, the W[1]-hardness of this variant of MSS parameterized by $k + \omega$ follows directly from a known reduction (Ganian, Klute, and Ordyniak 2021, Lemma 3).

Theorem 9. HDG-NASH *and* HDG-INDIVIDUAL *are* W[1]*-hard parameterized by* $\gamma + \rho_{\geq 1} + \tau$.

Proof Sketch. Given an instance $\mathcal{I} = (S_1, \ldots, S_\omega, t, k)$ of partitioned MSS, our reduction constructs an instance of HDG as follows. For each $i \in [\omega]$, we introduce one *marker* agent with color m_i (a *marker* color) and for each $j \in [k]$ we introduce $t[j]$-many (where $t[j]$ is the value of t on the j-th coordinate) normal agents, all with color g_j; collectively, we refer to all the former agents as *markers* and all the latter agents as *normal*. Intuitively, the k marker agents will each model the selection of one vector in S' and the normal agents represent the numbers in individual coordinates. The total number of colors is $k + \omega$ and we set[3] $\rho_{\geq 1}$ to ω.

Formally, the preferences of the agents are as follows. Marker agents have trichotomous preferences and each have their own agent type. In particular, a coalition C is *satisfying* for the marker agent with color m_i, $i \in [\omega]$ if: (1) C contains no other marker color than m_i and (2) there exists $s \in S_i$ such that for each $j \in [k]$, the number of agents of color g_j in C is precisely $s[j]$. Condition (2) is formalized via setting the palette to $\left(\frac{s[j]}{1+\sum_{j' \in [k]} s[j']}\right)$ for each color g_j. Marker agents strictly prefer being in a satisfying coalition than being alone (i.e., in a coalition containing only a single color m_i), and strictly prefer being alone than any other non-satisfying coalition.

Normal agents all have the same agent type and also have trichotomous preferences: they strictly prefer being in a coalition containing precisely only one marker color m_i to being in a coalition containing no marker color, and strictly prefer being in a coalition containing no marker color to being in a coalition containing more than one marker color. This concludes the description of our instance $\mathcal{I}'$ of HDG. Observe that $\tau = 1 + \omega$. The construction can be carried out in polynomial time. To complete the proof, it now remains to show that $\mathcal{I}$ is a Yes-instance if and only if $\mathcal{I}'$ is. □

[3]We remark that the proof can be extended to any value of $\rho_{\geq 1} > \omega$ by using the trap gadget described above Claim 2.

The last lower bound required to complete our complexity picture is for the case parameterized by the size of coalitions, number of types, and the number of non-trivial coalitions.

Theorem 10. HDG-NASH *and* HDG-INDIVIDUAL *are* W[1]*-hard parameterized by* $\sigma + \tau + \rho_{\geq 2}$.

Proof Sketch. We design a parameterized reduction from the INDEPENDENT SET problem parameterized by the size of the solution. Let $(G = (V, E), k)$ be an instance of INDEPENDENT SET. There is a vertex agent a_v for every $v \in V$ with the (new) color v. There is a green (guard) agent G and a red and a blue agents that together constitute the trap gadget we built in the proof of Theorem 7. The approved coalitions $\mathcal{M}_G$ of the guard agent are all coalitions of size exactly $k + 1$ where this agent is in a coalition with any k vertex agents. Note that this is expressible using the ratios as there is a unique guard agent. The vertex agents are indifferent between any coalition of size $k + 1$ that contain the guard agent and do not contain any edge; if this is not possible; they only want to be alone. It is not hard to verify that this is a single master list, since vertex agents have unique colors. We note that these preference lists are large – in fact, they are as large as $\binom{|V|}{k}$ but we can encode them using a very simple oracle (with polynomial running time). We set $\rho_{\geq 2} = 1$ and observe that $\tau \leq 4$; this finishes the description of the instance of HDG and hence of the reduction. □

Parameterizing by the Smaller of Two Color Classes

Boehmer and Elkind (2020a) showed that the problem of finding a Nash stable outcome for Hedonic diversity game with 2 colors R, B is in XP when parameterized by $q = \min\{|R|, |B|\}$. Immediately after proving that result, they ask whether this problem is in FPT with respect to the same parameter. We answer this in the negative with a W[1]-hardness reduction.

For our reduction we start from a restricted version of the so called GROUP ACTIVITY SELECTION problem (Darmann et al. 2017), called the SIMPLE GROUP ACTIVITY SELECTION PROBLEM (sGASP). To distinguish terminology between the GROUP ACTIVITY SELECTION problem and our HDG setting, we use slightly different terminology for that problem.

We are given a set of *participants* P, a set of *activities* A and for each participant $p \in P$ a set of *approved group sizes* $S_p \subseteq \{(a, i) \mid a \in A, i \in [|P|]\}$ for the activities. The task is to decide whether there is an assignment π of participants to activities which is stable, i.e., whether there is π such that for every participant $p \in P$, $(\pi(p), |\pi^{-1}(\pi(p))|) \in S_p$. sGASP is known to be W[1]-hard parameterized by the number of activities (Eiben, Ganian, and Ordyniak 2018).

Theorem 11. HDG-NASH *restricted to instances with* $\gamma = 2$ *is* W[1]*-hard when parameterized by the size* q *of the smaller color class.*

Proof Sketch. Let $\mathcal{I}$ be an instance of sGASP with participants P, activities A and approved group sizes $(S_p)_{p \in P}$.

Moreover we consider an auxiliary sequence of integers $(z_i)_{i\in[|A|]}$ where simply $z_i = 100i + 1$.

We construct an instance $\mathcal{J}$ of HDG with the following agents:

- $|P|$ 'normal agents' which all have color *blue* and each of which corresponds to one participant in $\mathcal{I}$.
- $\sum_{i\in[|A|]} z_i$ 'marker agents' of color *red* such that for each $i \in [|A|]$, z_i of these marker agents have the same preferences and correspond to the i-th activity.
- $(400|A|^2)200|A|^2 + 1$ 'spoiler agents' which will have the same preferences and all have color *red*.

Note that in this way the number of red agents is polynomial in $|A|$, the parameter of the W[1]-hard problem SGASP.

Let the agents' preferences be given as follows: For each normal agent j corresponding to a participant p in $\mathcal{I}$, we let red ratios in coalitions be preferred in the order

$$\left\{\frac{z_i}{z_i+t}\,\middle|\,(a,t)\in S_p \wedge a \text{ is the } i\text{-th activity}\right\} \succ_j 0 \succ_j \dots$$

Intuition: The number of red agents in a coalition encodes which activity the participants corresponding to the blue agents in that coalition should be assigned to. Then the preferences of the normal agents ensure that at least in terms of their ratio, the resulting group sizes behave like the approved group sizes would.

For each marker agent j corresponding to the i-th activity in $\mathcal{I}$, we let red ratios in coalitions be preferred in the order

$$\left\{\frac{z_i}{z_i+2s(|A|+1)-1},\dots,\frac{z_i}{z_i+2s|A|+1}\right\} \succ_j 1 \succ_j \dots$$

where s is an integer associated with the special variant of SGASP used here.

Intuition: For the preferences of the normal agents to take into account *all* participants corresponding to blue agents which will be assigned to the same activity we have to ensure that all the blue agents that will be assigned to the same activity will be in the same coalition. The marker agents should identify which unique coalition will correspond to which activity. Correspondingly their preferences allow to be in the same coalition as the marker agents associated with the same activity and an arbitrary number of normal agents between $2s|A| + 1$ and $2s(|A| + 1) - 1$.

For each spoiler agent j, we let red ratios in coalitions be preferred in the order $\mathfrak{B} \succ_j \mathfrak{S} \setminus \mathfrak{B} \succ_j 1 \succ_j \dots$, where

$$\mathfrak{B} = \left\{\frac{1}{1+b}\,\middle|\, b\in\mathbb{N}\right\} \text{ and}$$

$$\mathfrak{S} = \left\{\frac{r}{r+b}\,\middle|\, \exists i\in[|A|],\ \exists t\in\{2s|A|+1,\dots,2s(|A|+1)-1\} \text{ odd } \frac{r}{r+b}=\frac{z_i}{z_i+t}\wedge r\le 75i+1\right\}$$

Intuition: The spoiler agents can in some sense be regarded as the crux of our reduction. They will ensure that (1) there is no stable outcome containing a coalition with blue agents only, and (2) every stable outcome contains exactly one coalition for each activity and the marker agents for each activity are not distributed in multiple coalitions.

Claim 3. *If $\mathcal{I}$ is a* Yes*-instance, then $\mathcal{J}$ is also a* Yes*-instance.*

For the converse direction we assume that $\mathcal{J}$ is a Yes-instance and fix a solution Π that witnesses this. In the following claims we always speak about coalitions in Π.

Claim 4. *Every normal agent j with corresponding participant p in $\mathcal{I}$ is in a coalition with red ratio $\frac{z_i}{z_i+t}$ for some i and t for which a is the i-th activity in $\mathcal{I}$ and $(a,t)\in S_p$.*

Claim 5. *Whenever a set of normal agents $J = \{j_1,\dots,j_\ell\}$, corresponding to participants $p_1,\dots,p_\ell$, respectively, are together in the same coalition then the red ratio of the coalition is $\frac{z_i}{z_i+t}$ for some i and t for which a is the i-th activity in $\mathcal{I}$ and $(a,t)\in\bigcap_{k\in[\ell]} S_{p_k}$.*

Claim 6. *For every $i\in[|A|]$ there is at most one coalition with normal agents whose red ratio is of the form $\frac{z_i}{z_i+t}$ with $(a,t)\in S_p$ for every normal agent in that coalition and its corresponding participant $p\in P$ where a is the i-th activity.*

Finally we can show that $\mathcal{I}$ and $\mathcal{J}$ are indeed equivalent.

Claim 7. *If $\mathcal{J}$ is a* Yes*-instance, then $\mathcal{I}$ is also a* Yes*-instance.*

To prove Claim 7 and thus conclude the proof of Theorem 11, we proceed as follows. For $i\in[|A|]$, using Claim 6, let C_i be the unique coalition in Π (if there is any) with red ratio of the form $\frac{z_i}{z_i+t}$ with $(a,t)\in S_p$ for every normal agent in C_i and its corresponding participant $p\in P$ where a is the i-th activity. We define $\pi(p)=a$ such that the agent corresponding to p is in coalition C_i in Π and a is the i-th activity in A. By Claim 4 this defines π on all elements of P. Moreover Claim 5 implies that π is a solution for $\mathcal{I}$. ☐

Concluding Remarks

In their recent work, Boehmer and Elkind (2020a) also introduced a restriction to Hedonic diversity games where an agent i's preferences only depend on the ratio between i's color and the size of the coalition—in other words, which other colors occur in the coalition does not matter for i. Computing stable outcomes for these "*own-color*" Hedonic diversity games (hereinafter OWN-HDG) is a special case of the general HDG problem considered here, notably the case where the preferences of agents only depend on one element of the palette. As a consequence, every algorithmic result obtained for HDG in this paper immediately carries over also to OWN-HDG; however, the same is not true for algorithmic lower bounds. In fact, as our final result we give a concrete example where the complexity of HDG differs from that of OWN-HDG: While HDG is NP-hard when restricted to instances with $\rho_{\geq 2} = 2$ (and even $\rho_{\geq 1} = 3$) as per Theorem 8, parameterizing by the number of coalitions yields a non-trivial dynamic-programming based algorithm for OWN-HDG-NASH.

Theorem 12. OWN-HDG-NASH *is in* XP *parameterized by* $\rho_{\geq 2}$.

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