On the Distortion Value of the Elections with Abstention

Mohammad Ghodsi,[*†] **Mohamad Latifian,**[*] **Masoud Seddighin**[*]
[*]Sharif University of Technology, [†]Institute for Research in Fundamental Sciences (IPM) – School of CS
ghodsi@sharif.edu, {latifian, mseddighin}@ce.sharif.edu

Abstract

In Spatial Voting Theory, distortion is a measure of how good the winner is. It is proved that no deterministic voting mechanism can guarantee a distortion better than 3, even for simple metrics such as a line. In this study, we wish to answer the following question: how does the distortion value change if we allow less motivated agents to abstain from the election?

We consider an election with two candidates and suggest an abstention model, which is a more general form of the abstention model proposed by Kirchgässner (2003). We define the concepts of the *expected winner* and the *expected distortion* to evaluate the distortion of an election in our model. Our results fully characterize the distortion value and provide a rather complete picture of the model.

1 Introduction

The goal in Social Choice Theory is to design mechanisms that aggregate agents' preferences into a collective decision. Voting is a well-studied method for aggregating preferences which has numerous applications in multi-agent systems. Roughly, a voting mechanism takes the preferences of the agents over a set of alternatives as input and selects one of the alternatives as the winner.

One approach to estimate the quality of a voting mechanism is to use the utilitarian view which assumes that each agent has cost over the alternatives. For example, spatial models, locate the voters and the alternatives in a finite metric space $\mathcal{M}$, and the cost of alternative X for voter v_i is equal to their distance. Considering these costs, the optimal candidate is defined to be the candidate that minimizes the social cost. Ideally, we would like the optimal candidate to be the winner; however, since voting mechanisms only take the ordinal preferences of voters, it is reasonable to expect that the winner is not always optimal. The question then arises: how good is the winner, i.e., what is the worst-case ratio of the social cost of the winner to the social cost of the optimal candidate? This ratio is called the distortion value of a voting mechanism. It is known that no deterministic voting mechanism can guarantee a distortion better than 3, even for simple metrics such as a line. To see this, consider the example shown in Figure 1. In this example, candidate L is the optimal candidate, and under the plurality voting rule candidate R is the winner. Thus, the distortion value is

$$\frac{0.51(0.5-\varepsilon)+0.49\cdot 1}{0.51(0.5+\varepsilon)} \simeq 3.$$

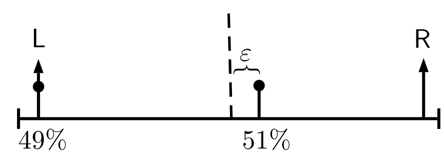

Figure 1: An example with distortion value 3.

The example of Figure 1 shows the lower-bound of 3 on the distortion value. However, it seems unrealistic in some ways. Although the voters located near the point 0.5 are closer to R, they have a very low incentive to vote for R, since their costs for both candidates are almost equal. On the other hand, agents located at 0 have a strong motivation to vote for L. In fact, if voters are allowed to abstain (which is a natural assumption in many real-world elections, especially in large elections), we expect L to be the winner rather than R.

In this study, our goal is to tackle this problem: how does the distortion value change if we allow less motivated agents to abstain from voting?

1.1 Abstention

Scientists have long studied the factors affecting participation in an election. In particular, Wolfinger and Rosenstone (1999) argue that better educated and more informed voters participate with a higher probability, or Lijphart (1997) discusses that the voters on the left side of the political spectrum participate less frequently than the ones on the right. Similarly, the decision to vote may rely on variables such as income level or the sense of civic duty (Wolfinger and Rosenstone 1980).

Traditionally, both game theoretic and decision theoretic models of turnout have been proposed. In the heart of most of these models lies the assumption that there are costs for voting [1]. These costs include the costs of collecting and pro-

[1]There are other decision theoretic explanations of abstention that do not rely on costs, e.g., see (1997).

cessing information, waiting in the queue and voting itself. Presumably, if a voter decides to abstain, she does not have to pay these costs. Thus, a rational voter must receive utility from voting. There is evidence suggesting that voters behave strategically when deciding to vote and take the costs and benefits into account. For example, Blais (2000) finds that the closeness of elections influences the turnout; or Riker and Ordeshook (1968) show that the turnout is inversely related to voting costs.

Apart from social-psychological traits, spatial models of voting suggest that voters' abstention may stem from their ideological distances from the candidates. The work of Downs (1957) initiated this line of research. He argues that in a two-candidate election under the majority rule, the choice between voting and abstaining is related to the voter's comparative evaluation of the candidates. Riker and Ordeshook (1968) improve this model by reformulating the original equation to incorporate other social and psychological factors.

Many empirical studies in the spatial theory of abstention frequently suggest that citizens are more likely to abstain when they feel indifferent toward the candidates or alienated from them. However, the models introduced by Downs (1957) and Riker and Ordeshook (1968) are only capable of representing the indifference-based abstention which occurs when the difference between the costs of candidates for a voter is too small to justify voting costs. On the other hand, these models cannot justify alienation-based abstention, which occurs when a voter is too distant from the alternatives to justify voting costs. To alleviate this, some studies argue that the relative distance plays a more critical role than the absolute distance (Kirchgässner 2003; Geys 2006).

1.2 Our Work

In this paper, we wish to study the effect of abstention on the distortion value. To represent abstention, we consider a simple spatial model, which is a more general form of the model suggested by Kirchgässner (2003). In this model, there are two candidates, and the voters decide whether to vote or abstain based on a comparison between the cost (i.e., distance) of their preferred alternative and the cost of the other alternative. We define the concepts of *expected winner* and *expected distortion* to evaluate the distortion of an election in our model. Our results fully characterize the distortion value and provide a complete picture of the model. For the special case that our model conforms exactly to that of (Kirchgässner 2003), we show that the distortion of the expected winner is upper bounded by 1.522. We also give an almost tight upper bound on the expected distortion value of big elections.

Finally, we generalize our results to include arbitrary metric spaces. We show that the same upper bounds we obtained on the distortion value for the line metric also work for arbitrary metric space.

1.3 Related Work

The utilitarian view, which assumes all of the voters have costs for each alternative, is a well-studied approach in welfare economics (Roemer 1998; Ng 1997). This viewpoint has received attention from the AI community during the past decade (Procaccia and Rosenschein 2006; Boutilier et al. 2015). Procaccia and Rosenschein (2006) first introduced distortion as a benchmark for measuring the efficiency of a social choice rule in utilitarian settings. The worst-case distortion of many social choice functions is shown to be high. However, imposing some mild constraints on the cost functions yields strong positive results. One of these assumptions, which is reasonable in many political and social settings, is the spatial assumption, which assumes that the agent costs form a metric space. (Enelow and Hinich 1984; Merrill and Grofman 1999).

Anshelevich, Bhardwaj and Postl (2015) were first to analyze the distortion of ordinal social choice functions when evaluated for metric preferences. For plurality, Borda, and k-approval rules, they prove that the worst case distortion is unbounded. On the positive side, they show that for the Copeland rule, the distortion value is at most 5. They also prove the lower bound of 3 for any deterministic mechanism. They conjecture that the worst-case distortion of Ranked Pairs social choice rule meets this lower-bound, which is later refuted by Goel, Krishnaswamy, Anilesh, and Munagala (2017).

Anshelevich and Postl (2017) consider randomized social choice rules. The output of such rules is a probability distribution over the set of alternatives rather than a single winning alternative. They show that for α-decisive metric spaces [2] any randomized rule has a lower-bound of $1 + \alpha$ on the distortion value. For the case of two alternatives, they propose an optimal algorithm with the expected distortion of at most $1+\alpha$. Cheng et al. (2017b) characterized the positional voting rules with constant expected distortion value (independent of the number of candidates and the metric space). Cheng et al. (2017a) consider the setting that the candidates are drawn independently from the population of voters and prove the tight bound of 1.1716 for the distortion value in a line metric and an upper-bound of 2 for an arbitrary metric space.

In addition to the studies mentioned in Section 1.1, many other studies consider the effects of abstention in various types of elections. For example, many voting mechanisms observe a common flaw, which states that a voter may obtain a better outcome by not voting (Fishburn and Brams 1983). This phenomenon is known as the no-show paradox. In a seminal paper, Moulin (1988) shows that every Condorcet consistent method is susceptible to the no-show paradox. Desmedt and Elkind in (2010) propose a game theoretic analysis of the plurality voting with the possibility of abstention and characterize the preference profiles that admit a pure Nash equilibrium. Rabinovich et al. (2015) consider the computational aspects of iterative plurality voting with abstention.

[2] A metric space is α-decisive, if for every voter, the cost of her preferred choice is at most α times the cost of her second best choice.

2 Preliminaries

Every election $\mathbb{E}$ consists of two ingredients: a set $\mathcal{V}$ of n voters and a set $\mathcal{C}$ of two candidates. We denote the i'th voter by v_i, and the candidates by L and R. Furthermore, we suppose that the candidates and the voters are embedded in a finite metric space $\mathcal{M}$. Unless explicitly stated otherwise, we suppose that $\mathcal{M}$ is a line, and L and R are located at points 0 and 1, respectively. For each voter v_i, we refer to her location by x_i. Moreover, we denote by $d_{i,\mathsf{X}}$, the distance between voter v_i and candidate $\mathsf{X} \in \{\mathsf{L}, \mathsf{R}\}$.

Definition 2.1. *For every candidate $\mathsf{X} \in \{\mathsf{L}, \mathsf{R}\}$, we define the average social cost of X as*

$$cost(\mathsf{X}) = \frac{\sum_i d_{i,\mathsf{X}}}{n}.$$

The optimal candidate is the candidate that minimizes the average social cost, i.e.,

$$\arg\min_{\mathsf{X}\in\mathcal{C}} cost(\mathsf{X}).$$

We consider a simple scenario where the winner is elected via the majority rule (in case of a tie, the winner is determined by tossing a fair coin). Furthermore, we suppose that each voter either abstains or votes for one of the candidates. In Section 2.1 we give a formal description of the voting behavior of the agents.

2.1 Voting Behavior of individuals

As mentioned, we employ a simple probabilistic model, where each voter independently decides whether to abstain or participate by evaluating her distances from the candidates. Fix a voter v_i and let X be the candidate who is closer to v_i in $\mathcal{M}$ and $\bar{\mathsf{X}}$ be the other candidate. We suppose that v_i votes sincerely for her preferred candidate X with a probability p_i, where p_i is a function of $d_{i,\mathsf{X}}$ and $d_{i,\bar{\mathsf{X}}}$, and abstains with probability $1 - p_i$.

Denote by f the probability function that p_i is derived from, i.e., $p_i = f(d_{i,\mathsf{X}}, d_{i,\bar{\mathsf{X}}})$. Since f represents the probability of voting, we expect f to satisfy certain axiomatic assumptions. Recall that in the spatial voting models, there are two crucial sources of abstention:

i **Indifference-based Abstention (IA):** the smaller the difference between the distances of a voter from the candidates is, the less likely it is that she casts a vote.

ii **Alienation-based Abstention (AA):** the further a voter is located from the candidates, the less likely it is that she casts a vote.

To illustrate, for the voters in Figure 2, we have:

- Voters v_1, v_2, and v_3 prefer L and voters v_5 and v_6 prefer R.
- Voter v_1 has a strong incentive to cast a vote since her cost for L is zero.
- Voter v_4 always abstains, since her costs for both the candidates are equal (IA).
- For voters v_2 and v_3, we have $p_2 \geq p_3$, since $d_{2,\mathsf{L}} \leq d_{3,\mathsf{L}}$, and $d_{2,\mathsf{R}} - d_{2,\mathsf{L}} \geq d_{3,\mathsf{R}} - d_{3,\mathsf{L}}$ (IA,AA).

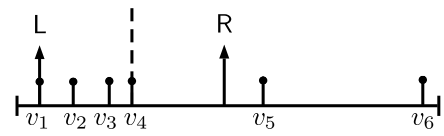

Figure 2: A simple election instance.

- For voters v_5 and v_6, we have $p_5 \geq p_6$, since v_6 is more alienated (AA).

As mentioned, the models of Downs (1957) and Riker and Ordeshook (1968) are only capable of explaining the Indifference-based abstention, since they only consider the absolute difference between the distances of the candidates to a voter. To resolve this, some recent studies argue that the relative distance, rather than absolute distance, is relevant.

In this study, we follow the model of Kirchgässner (2003) which is based on the relative distances. The idea behind their model is that, the probability that a voter casts a vote depends on her ability to distinguish between the candidates. By Weber–Fechner's law (see (Fechner 2012)), the ability to distinguish between the candidates depends on their relative distance to the voter. Formally, the probability p_i that voter v_i votes for X is calculated by the following formula:

$$p_i = f(d_{i,\mathsf{X}}, d_{i,\bar{\mathsf{X}}}) = \frac{|d_{i,\mathsf{X}} - d_{i,\bar{\mathsf{X}}}|}{d_{i,\bar{\mathsf{X}}} + d_{i,\mathsf{X}}}. \tag{1}$$

In this paper, we consider a more general form of Equation (1) as follows:

$$p_i = \zeta_\beta(d_{i,\mathsf{X}}, d_{i,\bar{\mathsf{X}}}) = \left(\frac{|d_{i,\mathsf{X}} - d_{i,\bar{\mathsf{X}}}|}{d_{i,\bar{\mathsf{X}}} + d_{i,\mathsf{X}}}\right)^\beta, \tag{2}$$

where β is a constant in $[0, 1]$. Figure 3 shows the behavior of ζ_β for different values of β. As is clear from the figure, for the smaller values of β, voters are more eager to participate. In fact, the exponent β can be seen as a quantitative measure of how much this ideological distance matters. For the special case of $\beta = 0$, voters always participate in the election, regardless of their location.

It can be easily observed that for any $0 \leq \beta \leq 1$, function ζ_β satisfies all the desired criteria.

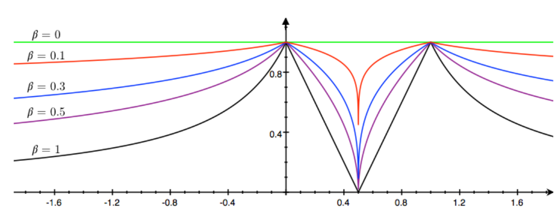

Figure 3: ζ_β for different values of β.

2.2 The Expected Winner and the Expected Distortion

As discussed, our assumption is that the majority rule determines the winner. However, considering the stochastic behavior of the voters, the winner is not deterministic, i.e., each candidate has a probability of winning. Denote by $\#\mathsf{X}$, the expected number of voters who vote for X. Furthermore, denote by P_X, the probability that candidate X wins the election. We define the **expected winner** of the election as the candidate with the maximum expected number of votes.

Definition 2.2. *For a candidate $\mathsf{X} \in \{L, R\}$, we define the distortion of X, denoted by $D(\mathsf{X})$ as the ratio $cost(\mathsf{X})/cost(\mathsf{O})$, where $\mathsf{O} \in \{L, R\}$ is the optimal candidate.*

Note that the distortion value of the optimal candidate is 1. We consider two approaches to evaluate the distortion of an election $\mathbb{E}$. In the first approach, we define the distortion of $\mathbb{E}$ as the distortion of the expected winner, i.e., $D(\mathsf{W})$, where $\mathsf{W} \in \{\mathsf{L}, \mathsf{R}\}$ is the expected winner. Another approach, which is more promising, is to define the distortion of $\mathbb{E}$ as the expected distortion of the winner, over all possible outcomes, i.e.,

$$\mathsf{P}_\mathsf{L} \cdot D(\mathsf{L}) + \mathsf{P}_\mathsf{R} \cdot D(R). \tag{3}$$

For convenience, we use $\dot{D}(\mathbb{E})$ to refer to the distortion value of $\mathbb{E}$ regarding the former approach and $\ddot{D}(\mathbb{E})$ to refer to the distortion value regarding the latter.

We dedicate two separate sections to analyze the distortion value of elections according to both approaches. Even though the maximum distortion depends on the value of β, we provide essential tools to analyze the election for any $\beta \in [0, 1]$.

3 Distortion of the Expected Winner

In this section, we analyze the distortion value of the expected winner. Recall that the expected winner is the candidate with a higher expected number of votes. There are two reasons why we consider the distortion value of the expected winner. First, since the number of votes each candidate receives is concentrated around its expectation, in elections with a large number of voters (relative to the number of candidates), the expected winner has a very high chance of winning, especially, when there is a non-negligible difference between the expected number of votes each candidate receives. Secondly, we use the tight upper-bound on the distortion value of the expected winner to prove an upper bound on the expected distortion of the election for the second approach.

Throughout this section, we assume that the probability that a voter casts a vote is

$$\zeta_\beta = \left(\frac{|d_{i,X} - d_{i,\bar{X}}|}{d_{i,\bar{X}} + d_{i,X}} \right)^\beta .$$

Furthermore, we suppose w.l.o.g. that L is the expected winner. Moreover, we assume that the optimal candidate is R, otherwise the distortion equals 1. We also consider four regions A, B, C and D as illustrated in Figure 4.

In Theorem 3.1 we state the main result of this section.

Theorem 3.1. *There exists an election $\mathbb{E}$, such that $\dot{D}(\mathbb{E})$ is maximum, and the voters in $\mathbb{E}$ are located in at most three different locations.*

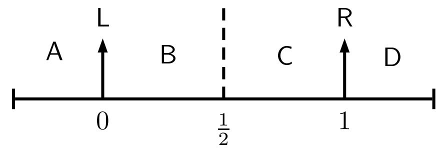

Figure 4: Regions A, B, C, and D

In the interest of the space, here, we only present a sketch of the proof of Theorem 3.1 and leave the details to the full version of the paper [3] . The basic idea to prove Theorem 3.1 is as follows: we prove that for every election $\mathbb{E}$, there exists an election instance $\mathbb{E}'$ with the same expected winner, and $\dot{D}(\mathbb{E}') \geq \dot{D}(\mathbb{E})$. Also, the voters in $\mathbb{E}'$ are located in at most 3 different locations. To show this, we collect the voters in $\mathbb{E}$ by carefully moving them forward and backward via a sequence of *valid displacements*, as defined in Definition 3.2.

Definition 3.2. *For an election $\mathbb{E}$, define a displacement as the operation of moving a subset of the voters forward or backward on the line to a new location. A displacement is valid if it does not alter the expected winner, and furthermore, does not decrease the distortion value of the expected winner.*

In Lemmas 3.3,3.4, and 3.5 we introduce three sorts of valid displacement which help us collect the voters. Figure 5, illustrates a summary of these displacements.

Lemma 3.3. *Moving a voter v_i from $x_i \in A$ to 0 is a valid displacement.*

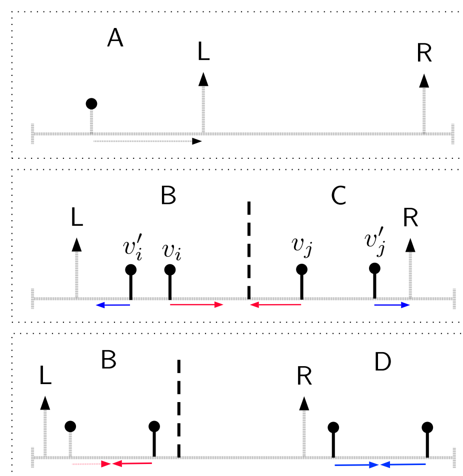

Figure 5: Valid displacements introduced in Lemmas 3.3, 3.4, 3.5.

Lemma 3.4. *Consider voters v_i and v_j located respectively at $x_i \in B$ and $x_j \in C$. Then,*

[3]The full version of the paper is available at http://ce.sharif.edu/ latifian/files/AAAI19-Full.pdf

- *If $d_{i,\mathsf{L}} \leq d_{j,\mathsf{R}}$, moving v_i to $x_i + x_j - 1/2$ and v_j to $1/2$ is a valid displacement.*
- *If $d_{i,\mathsf{L}} > d_{j,\mathsf{R}}$, moving v_i to $x_i - 1 + x_j$ and v_j to 1 is a valid displacement.*

Lemma 3.5. *Consider voters v_i, v_j, where $x_i, x_j \in B \cup D$ and both v_i and v_j belong to the same region. Then, moving both the voters to $\frac{x_i + x_j}{2}$ is a valid displacement.*

Using these three types of valid displacements, we can establish an election with the maximum distortion, and the following structure (see Figure 6): the interior of regions A and C contain no voter. All the voters are located at three points, namely x_b, x_d and x_m, where $x_b \in \mathsf{B}$, $x_m = 1/2$, and $x_d \in \mathsf{D}$. Note that, the maximum distortion value and the location of x_b and x_d in the worst-case scenario essentially relies on the value of β in ζ_β.

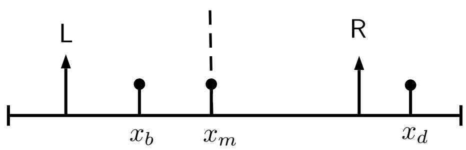

Figure 6: For any $\beta \in [0, 1]$, there is an election with the maximum distortion, and the above structure.

3.1 A Tight Upper Bound on $\dot{D}(\cdot)$

In this section, we discuss the value of $\dot{D}(\mathbb{E})$, when the probability function is ζ_β and $0 \leq \beta \leq 1$.

Let us first consider one of the boundary cases: ζ_0. For ζ_0, the probability that a voter casts a vote is

$$\left(\frac{|d_{i,X} - d_{i,\bar{X}}|}{d_{i,\bar{X}} + d_{i,X}}\right)^0 = 1,$$

independent of her location. Therefore, the same example demonstrated in Section 1 implies that no upper-bound better than 3 can be obtained for $\dot{\mathsf{D}}(\mathbb{E})$, when the participation function is ζ_0.

Now, consider ζ_β, and let $\mathbb{E}$ be the election that maximizes $\dot{\mathsf{D}}(\mathbb{E})$. As discussed in the previous section, we can assume w.l.o.g that the voters in $\mathbb{E}$ are located at three points, namely, $x_b \in \mathsf{B}, x_d \in \mathsf{D}$, and $x_m = 0.5$. Suppose that the ratio q_b of the voters are at x_b, the ratio q_d of the voters are at x_d, and the ratio q_m of the voters are at $1/2$ ($q_b + q_d + q_m = 1$). We have

$$\#\mathsf{L} = (1 - 2x_b)^\beta q_b n \quad \text{and} \quad \#\mathsf{R} = \left(\frac{1}{(2x_d - 1)^\beta}\right) q_d n.$$

Since L is the expected winner, we have

$$(1 - 2x_b)^\beta q_b n \geq \left(\frac{1}{(2x_d - 1)^\beta}\right) q_d n.$$

On the other hand, we have

$$\mathsf{cost}(\mathsf{L}) = q_b x_b + q_d x_d + q_m/2,$$

and

$$\mathsf{cost}(\mathsf{R}) = q_b(1 - x_b) + q_d(x_d - 1) + q_m/2.$$

Thus,

$$\begin{aligned}
\dot{D}(\mathbb{E}) &= \frac{\mathsf{cost}(\mathsf{L})}{\mathsf{cost}(\mathsf{R})} \\
&= \frac{q_b x_b + q_d x_d + q_m/2}{q_b(1 - x_b) + q_d(x_d - 1) + q_m/2} \\
&= \frac{q_b x_b + (1 - q_b - q_m) x_d + q_m/2}{q_b(1 - x_b) + (1 - q_b - q_m)(x_d - 1) + q_m/2}
\end{aligned}$$

Therefore, in order to find the maximum distortion, we need to solve the following optimization problem:

$$\begin{aligned}
\max \quad & \frac{q_b x_b + (1 - q_b - q_m) x_d + \frac{q_m}{2}}{q_b(1 - x_b) + (1 - q_b - q_m)(x_d - 1) + \frac{q_m}{2}} \\
\text{s.t.} \quad & (1 - 2x_b)^\beta q_b \geq \frac{1 - q_b - q_m}{(2x_d - 1)^\beta} \\
& 0 \leq q_b, q_m \leq 1 \\
& q_m + q_b \leq 1 \\
& 0 \leq x_b \leq 1/2 \\
& 1 \leq x_d.
\end{aligned} \tag{4}$$

Now consider another boundary case: ζ_1. For ζ_1 the answer to the above optimization problem is $\frac{(1+\sqrt{2})^2}{1+2\sqrt{2}} \simeq 1.522$, which can be obtained by choosing $q_b = \frac{1}{2+\sqrt{2}}$, $q_m = 0$, $x_b = 0$, and $x_d = \frac{2+\sqrt{2}}{2}$. A graphical representation of this construction is shown in Figure 7.

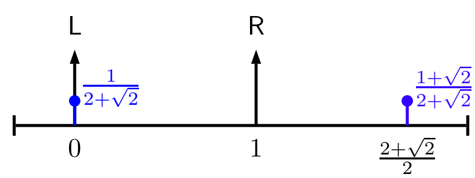

Figure 7: A tight example for $\beta = 1$.

In general for $0 < \beta < 1$, the maximum distortion value equals the answer of Optimization Problem (4).

In Figure 8, we show the answer of this program for different values of β. As illustrated in Figure 8, It can be seen that the minimum possible distortion value is $\simeq \sqrt{2}$. (for $\beta \simeq 0.705$.)

4 Expected Distortion

Recall that in our second approach, we define the distortion of an election as the expected distortion of the winner, where the expectation is taken over random behaviors of the voters. Throughout this section, we assume that the probability that a voter casts a vote is

$$\zeta_\beta = \left(\frac{|d_{i,X} - d_{i,\bar{X}}|}{d_{i,\bar{X}} + d_{i,X}}\right)^\beta.$$

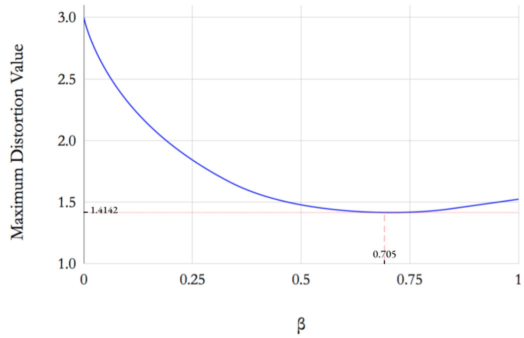

Figure 8: Maximum distortion value for $\zeta_\beta (0 \le \beta \le 1)$.

Furthermore, we suppose w.l.o.g that candidate R is the optimal candidate. Thus, Equation (3) can be rewritten as

$$\ddot{D}(\mathbb{E}) = \mathsf{P}_\mathsf{L} \frac{\mathsf{cost}(L)}{\mathsf{cost}(R)} + \mathsf{P}_\mathsf{R}. \tag{5}$$

In Theorem 4.1, we prove that there is an election with the maximum distortion value and a simple structure.

Theorem 4.1. *There exists an election $\mathbb{E}$, such that $\ddot{D}(\mathbb{E})$ is maximum, and the voters in $\mathbb{E}$ are located in at most four different locations.*

The basic idea to prove Theorem 4.1 is similar to that of Theorem 3.1; we collect the voters using a sequence of valid displacements, albeit with a new definition of valid displacement.

Definition 4.2. *A displacement is valid, if it does not decrease $\ddot{D}(\mathbb{E})$.*

The procedure of proving that a displacement is valid for this case is relatively tougher than the previous model. The reason is that we do not have a closed-form expression which represents the winning probability of each candidate. In Lemmas 4.3, 4.4, and 4.5 we explain our tools to discover valid displacements.

Lemma 4.3. *For each voter $v_i \in$ A, there is a point $x_i \in$ B such that moving v_i to x_i is a valid displacement. Furthermore, for each voter $v_j \in$ C, there is a point $y_j \in$ D such that moving v_j to y_j is a valid displacement.*

Lemma 4.4. *Let v_i and v_j be two voters located respectively at x_i and $x_j \in$ B. Furthermore, suppose that*

$$\varepsilon \le x_i \le x_j \le 1/2 - \varepsilon,$$

where ε is a positive constant. Then, at least one of the following displacements is valid:

- *Moving both the voters to $\frac{x_i + x_j}{2}$.*
- *Moving v_i to $x_i - \varepsilon$ and v_j to $x_j + \varepsilon$.*

Lemma 4.5. *Let v_i and v_j be two voters located respectively at $x_i, x_j \in$ D. Then, there exists a point x between x_i and x_j, such that moving both the voters to x is a valid displacement.*

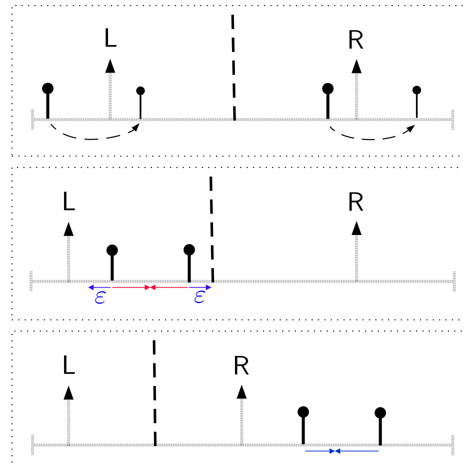

Figure 9: Valid displacements introduced in Lemmas 4.3, 4.4, 4.5.

Figure 9, shows a summary of the displacements described in Lemmas 4.3, 4.4 and 4.5 is illustrated. Using these displacements, one can establish an election with the maximum distortion, and the following structure (see Figure 10): the interior of regions A and C contain no voter. All the voters are located at four different points, namely x_l, x_b, x_m, and x_d, where $x_l = 0$, $x_b \in$ B, $x_m = 1/2$, and $x_d \in$ D. The maximum distortion value and the exact location of x_b and x_d in the worst-case scenario essentially relies on the value of β in ζ_β.

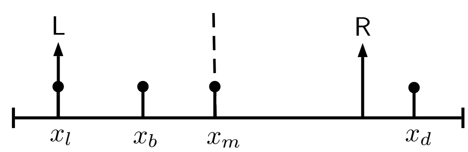

Figure 10: For any $\beta \in [0, 1]$, there is an election with the maximum distortion and this structure.

4.1 An Almost Tight Bound on $\ddot{D}(\cdot)$

In this section, we discuss the value of $\ddot{D}(\cdot)$, when the probability function is ζ_β with $\beta \in [0, 1]$. As mentioned, to prove our upper and lower bounds in this section, we subsequently use the bounds obtained in Section 3.1. For brevity, we use $\dot{D}^*_\beta$ to refer to the maximum distortion value of the expected winner for probability function ζ_β.

Similar to Section 3.1, we begin with the boundary case of ζ_0. By a similar argument as in Section 3.1, for ζ_0 we have the tight bound of 3 on the value of maximum distortion. For $\beta > 0$, we prove Theorem 4.6 which is the most technical result of the paper. This theorem provides an asymptotic upper bound on the maximum expected distortion for any ζ_β with $\beta \in (0, 1]$.

Theorem 4.6. *For any $\alpha > 0$ and $\beta \in (0, 1]$, the expected distortion value of every election whose candidates receive* *at least*

$$\frac{9(1+\sqrt{1+\alpha})(1+\alpha)}{8(2+\alpha-2\sqrt{1+\alpha})\alpha}$$

expected number of votes, is at most $(1+2\alpha)\dot{D}^*_\beta$.

For instance, for $\alpha = 1/6$, Theorem 4.6 states that for every election whose candidates receive at least 2552 expected number of votes, the distortion value is upper bounded by $4/3\dot{D}^*_\beta$ which for $\beta = 1$ is $4/3\dot{D}^*_1 \simeq 2$. We complement Theorem 4.6 by describing how to construct bad examples with distortion value near $\dot{D}^*$.

Example 1. *Consider Optimization Problem 4, with an additional constraint that* $\#\mathsf{L} \geq \#\mathsf{R}(1+\varepsilon)$*, and let* $\dot{D}^{**}$ *be the answer of this optimization problem and* $\mathbb{E}^{**}$ *be its corresponding election instance. By Chernoff bound, for large enough value of* $\#\mathsf{L}$*, candidate* L *almost surely wins the election, i.e.,*

$$\lim_{\#\mathsf{L}\to\infty} \dot{D}(\mathbb{E}^{**}) = D(\mathsf{L}) = \dot{D}^{**}.$$

Note that, the bound provided by Theorem 4.6 is almost tight; as the election size grows, the upper bounds of Theorem 4.6 tends to the distortion value of the Example 1. However, for elections with a small number of voters, the distortion value might be larger. For example, consider a simple scenario where there is one voter located at point $1+\varepsilon \in \mathsf{D}$ and $\beta = 1$ (see Figure 11). For this case, the distortion value is

$$\begin{aligned}\mathsf{P}_\mathsf{L}\cdot\frac{\mathsf{cost}(\mathsf{L})}{\mathsf{cost}(\mathsf{R})} + \mathsf{P}_\mathsf{R} &= \mathsf{P}_\mathsf{L}\cdot\frac{1+\varepsilon}{\varepsilon} + \mathsf{P}_\mathsf{R}\\ &= \frac{\varepsilon}{1+2\varepsilon}\cdot\frac{1+\varepsilon}{\varepsilon} + \frac{1+\varepsilon}{1+2\varepsilon}\\ &= \frac{2+2\varepsilon}{1+2\varepsilon},\end{aligned}$$

which tends to 2 as $\varepsilon \to 0$. We conjecture that this example is the worst possible scenario and the value of $\ddot{D}(\cdot)$ is upper bounded by 2 for any election with any size while $\beta = 1$.

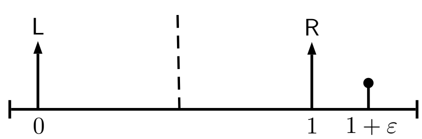

Figure 11: An example with maximum expected distortion. $\ddot{D}(E)$ tends to 2 as ε tends to 0.

5 General Metric

In this section, we generalize our results for the general metric space. Suppose that the voters and candidates are located in an arbitrary metric $\mathcal{M}$. By definition, for every voter i and candidates L, R we have:

- $d_{i,\mathsf{L}}, d_{i,\mathsf{R}} \geq 0$.
- $d_{i,\mathsf{L}} + d_{i,\mathsf{R}} \geq d_{\mathsf{L},\mathsf{R}}$ (triangle inequality).

We suppose without loss of generality that $d_{\mathsf{L},\mathsf{R}} = 1$. For this case, we prove Theorem 5.1, which extends our results to general metric spaces. Note that Theorem 5.1 considers both $\dot{D}(\cdot)$ and $\ddot{D}(\cdot)$. By this theorem for every $0 \leq \beta \leq 1$, the same upper bounds we obtained on the distortion value for the line metric also works for any arbitrary metric space.

Theorem 5.1. *Let* $D^{\mathcal{L}}_\beta$ *be the maximum distortion value for probability function* ζ_β *and line metric, and let* $D^{\mathcal{M}}_\beta$ *be the maximum distortion value for probability function* ζ_β *and arbitrary metric space* $\mathcal{M}$*. Then, we have* $D^{\mathcal{L}}_\beta \geq D^{\mathcal{M}}_\beta$.

6 Future Directions

In this study, we analyzed the distortion value in a spatial voting model with two candidates, where the voters were allowed to abstain. The set of results in this paper provides a rather complete picture of the model. However, the model we developed in this paper is for two candidate elections. Therefore, it does not consider the possible challenges that frequently arise in multi-candidate elections. One future direction is to extend this model to include multi-candidate elections.

Another interesting open question is to analyze the expected distortion value of the elections with a small number of voters. The counter-example in Section 4.1 refutes the existence of an upper bound better than 2. We believe that this example is the worst possible scenario. However, no formal proof is provided.

Acknowledgments

The authors would like to thank Ariel Procaccia for helpful discussion.

References

Anshelevich, E., and Postl, J. 2017. Randomized social choice functions under metric preferences. *Journal of Artificial Intelligence Research* 58:797–827.

Anshelevich, E.; Bhardwaj, O.; and Postl, J. 2015. Approximating optimal social choice under metric preferences. In *AAAI*, volume 15, 777–783.

Blais, A. 2000. *To vote or not to vote?: The merits and limits of rational choice theory*. University of Pittsburgh Pre.

Boutilier, C.; Caragiannis, I.; Haber, S.; Lu, T.; Procaccia, A. D.; and Sheffet, O. 2015. Optimal social choice functions: A utilitarian view. *Artificial Intelligence* 227:190–213.

Cheng, Y.; Dughmi, S.; and Kempe, D. 2017a. Of the people: voting is more effective with representative candidates. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, 305–322. ACM.

Cheng, Y.; Dughmi, S.; and Kempe, D. 2017b. On the distortion of voting with multiple representative candidates. *arXiv preprint arXiv:1711.07600*.

Desmedt, Y., and Elkind, E. 2010. Equilibria of plurality voting with abstentions. In *Proceedings of the 11th ACM conference on Electronic commerce*, 347–356. ACM.

Downs, A. 1957. An economic theory of political action in a democracy. *Journal of political economy* 65(2):135–150.

Enelow, J. M., and Hinich, M. J. 1984. *The spatial theory of voting: An introduction*. CUP Archive.

Fechner, G. T. 2012. *Elemente der psychophysik*, volume 2.

Feddersen, T. J., and Pesendorfer, W. 1999. Abstention in elections with asymmetric information and diverse preferences. *American Political Science Review* 93(2):381–398.

Fishburn, P. C., and Brams, S. J. 1983. Paradoxes of preferential voting. *Mathematics Magazine* 56(4):207–214.

Geys, B. 2006. 'rational'theories of voter turnout: a review. *Political Studies Review* 4(1):16–35.

Ghirardato, P., and Katz, J. N. 1997. Indecision theory: An informational model of roll-off.

Goel, A.; Krishnaswamy, A. K.; and Munagala, K. 2017. Metric distortion of social choice rules: Lower bounds and fairness properties. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, 287–304. ACM.

Kirchgässner, G. 2003. Abstention because of indifference and alienation, and its consequences for party competition: A simple psychological model. university of st. Technical report, Gallen Discussion Paper.

Lijphart, A. 1997. Unequal participation: Democracy's unresolved dilemma presidential address, american political science association, 1996. *American political science review* 91(1):1–14.

Merrill, S., and Grofman, B. 1999. *A unified theory of voting: Directional and proximity spatial models*. Cambridge University Press.

Moulin, H. 1988. Condorcet's principle implies the no show paradox. *Journal of Economic Theory* 45(1):53–64.

Ng, Y.-K. 1997. A case for happiness, cardinalism, and interpersonal comparability. *The Economic Journal* 107(445):1848–1858.

Procaccia, A. D., and Rosenschein, J. S. 2006. The distortion of cardinal preferences in voting. In *International Workshop on Cooperative Information Agents*, 317–331. Springer.

Rabinovich, Z.; Obraztsova, S.; Lev, O.; Markakis, E.; and Rosenschein, J. S. 2015. Analysis of equilibria in iterative voting schemes. In *AAAI*, volume 15, 1007–1013.

Riker, W. H., and Ordeshook, P. C. 1968. A theory of the calculus of voting. *American political science review* 62(1):25–42.

Roemer, J. E. 1998. *Theories of distributive justice*. Harvard University Press.

Wolfinger, R. E., and Rosenstone, S. J. 1980. *Who votes?*, volume 22. Yale University Press.