Strategy-Proof and Non-Wasteful Multi-Unit Auction via Social Network[1]

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Abstract

Auctions via social network, pioneered by Li et al. (2017), have been attracting considerable attention in the literature of mechanism design for auctions. However, no known mechanism has satisfied strategy-proofness, non-deficit, non-wastefulness, and individual rationality for the multi-unit unit-demand auction, except for some naïve ones. In this paper, we first propose a mechanism that satisfies all the above properties. We then make a comprehensive comparison with two naïve mechanisms, showing that the proposed mechanism dominates them in social surplus, seller's revenue, and incentive of buyers for truth-telling. We also analyze the characteristics of the social surplus and the revenue achieved by the proposed mechanism, including the constant approximability of the worst-case efficiency loss and the complexity of optimizing revenue from the seller's perspective.

1 Introduction

Auction theory has attracted much attention in artificial intelligence as a foundation of multi-agent resource allocation. One of the mainstreams in the literature is analyzing auctions from the perspective of mechanism design. In particular, several works studied how to design *strategy-proof* auctions, which incentivize each buyer to truthfully report her valuation function, regardless of the reports of the other buyers. One critical contribution in the literature is the development of the Vickrey-Clarke-Groves mechanism (VCG), which satisfies strategy-proofness and various other properties (Vickrey 1961; Clarke 1971; Groves 1973).

Li et al. (2017) proposed a new model of auctions, in which buyers are distributed in a social network and the information on the auction propagates over it. Utilizing a social network, the seller can advertise the auction to more potential buyers beyond her followers, as many works studied in network science (Emek et al. 2011; Borgatti et al. 2009; Jackson 2008; Kempe, Kleinberg, and Tardos 2003). From the buyers' perspective, however, forwarding the information increases the number of buyers, which reduces the possibility that they will get the item. Therefore, the main challenge in the auction via social network is how to incentivize buyers to forward the information to as many followers as possible, as well as truthfully reporting their valuation functions. For selling a single unit of an item, Li et al. (2017) developed an auction mechanism in which each buyer is incentivized to forward the information to her followers.

Zhao et al. (2018) studied a multi-unit unit-demand auction via social network, where each unit is identical and each buyer requires a unit. They proposed the generalized information diffusion mechanism (GIDM) and argued that it is strategy-proof. However, Takanashi et al. (2019) pointed out an error in their proof and argued that GIDM is not strategy-proof. They also proposed a strategy-proof mechanism for the same model, which however violates a revenue condition called *non-deficit*, i.e., the seller might suffer a deficit. To the best of our knowledge, for the multi-unit unit-demand auction via social network, no mechanism satisfying both strategy-proofness and non-deficit has been developed, except for some naïve ones.

The main objective of this paper is to propose a mechanism that satisfies both strategy-proofness and non-deficit, as well as some other properties. As Takanashi et al. (2019) pointed out, no mechanism satisfies those properties and Pareto efficiency, i.e., maximizing the social surplus, under certain natural assumptions. They thus considered weakening the non-deficit condition. In this paper, on the other hand, we consider a weaker efficiency property called *non-wastefulness*, which only requires the allocation of as many units as possible. Non-wastefulness has its own importance in practice. For example, in a spectrum auction, it is important to allocate as much frequency range as possible to carriers in order to guarantee a sufficient number of services.

We propose a new mechanism, called dinstance-based network auction mechanism for multi-unit, unit-demand buyers (DNA-MU), for a multi-unit unit-demand auction via social network, which satisfies strategy-proofness, non-deficit, non-wastefulness, and individual rationality, i.e., no buyer receives negative utility under truth-telling, and whose description is much simpler than GIDM. It is inspired by the concept of the *diffusion critical tree*, originally proposed in Li et al. (2017), which specifies, for each buyer i, the set of *critical* buyers for i's participation. If a buyer j is critical for

[1]A full version is available at http://arxiv.org/abs/1911.08809

another buyer i's participation, i.e., if i cannot participate in the auction without j's forwarding of information, j must receive a higher priority in the competition.

We then make a comprehensive comparison with two naïve mechanisms that also satisfy (most of) the above properties. One is based on VCG, being applied only to the buyers who are directly connected to the seller. The other mechanism simply allocates the units in the first-come-first-served manner with no payment. We show that the DNA-MU dominates both of these naïve ones in terms of social surplus and the seller's revenue. Furthermore, in those mechanisms, hiding the information, combined with reporting the true value, is also a dominant strategy, while this is not the case in our mechanism when $k \geq 2$. This indicates that each buyer has a stronger incentive for truth-telling in the DNA-MU.

We further analyze the characteristics of the social surplus and the revenue of the DNA-MU. About social surplus, it guarantees that each winner is in the set of top-k buyers except for her followers. It also has a constant worst-case efficiency loss, based on a measure proposed by Nath and Sandholm (2018), when an optimal reserve price is introduced. About revenue, we show that a revenue monotonicity condition fails and that maximizing revenue by optimally sending the information is NP-complete.

2 Preliminaries

We first define the standard notations for multi-unit unit-demand auctions. Let s be a seller who is willing to sell the set K of k identical units. Let N be the set of n buyers, where each buyer $i \in N$ has a unit-demand valuation function for K. Let $\boldsymbol{x} = (x_i)_{i \in N} \subset \{0, 1\}^n$ be an allocation, which specifies who obtains a unit, where $x_i = 1$ indicates that buyer i obtains a unit under allocation $\boldsymbol{x}$, and $x_i = 0$ otherwise. Let $v_i \in \mathbb{R}_{\geq 0}$ indicate the true unit-demand *value* of buyer i for a single unit. We assume that each buyer's utility is *quasi-linear*, i.e., the utility of buyer i under allocation $\boldsymbol{x}$, when she pays $p_i \in \mathbb{R}$, is given as $v_i \cdot x_i - p_i$.

Next, we define additional notations for the auction via social network. For each buyer $i \in N$, let $r_i \subseteq N \setminus \{i\}$ be the set of buyers to whom buyer i can forward the information, called i's *followers*. Also, let $r_s \subseteq N$ be the set of *direct buyers*, i.e., those to whom the seller s can directly send the information. Given $(r_i)_{i \in N \cup \{s\}}$, we define the auction network as a digraph $G = (N \cup \{s\}, E)$, where for each $i \in N \cup \{s\}$ and each $j \in r_i$, a directed edge, from i to j, is added to the set E. Note that r_i is also private information of buyer i in our model, so the auction network is defined according to *reported* $\boldsymbol{r}' = (r'_i)_{i \in N}$, where r'_i indicates the set of i's followers to whom i forwards the information. To summarize, for each i, the private information is given as $\theta_i = (v_i, r_i)$, called the true type of i, consisting of the true value v_i and the set r_i of the followers. Any reportable type $\theta'_i = (v'_i, r'_i)$ of i with true type $\theta_i = (v_i, r_i)$ satisfies $r'_i \subseteq r_i$, i.e., a buyer can only forward the information to her followers. Let $R(\theta_i)$ be the set of all reportable types by i with θ_i. Also, let $\boldsymbol{\theta}'$ denote the profile of types reported by all buyers and $\boldsymbol{\Theta}$ denote the set of all possible type profiles.

For notation simplicity, we introduce additional technical terms regarding the auction network. A buyer i is *connected* if a path $s \rightarrow \cdots \rightarrow i$ in G is formed based on the reported $\boldsymbol{r}'$. Let $\hat{N}$ denote the set of connected buyers. For each i, let $d(i)$ denote the distance of the shortest path from s to i. If i is not connected, we assume $d(i) = \infty$. Given $\boldsymbol{\theta}'$, a buyer $j \in \hat{N}$ is a *critical parent of* $i \in \hat{N}$ if, without j's participation, i is not connected, i.e., j appears in *any* path from s to i in G. Let $P_i(\boldsymbol{\theta}') \subseteq \hat{N}$ denote the set of all critical parents of i under $\boldsymbol{\theta}'$. The buyer $j \in P_i(\boldsymbol{\theta}')$ closest to i is called the *least critical parent of* i. An allocation $\boldsymbol{x}$ is *feasible* if $\sum_{i \in N} x_i \leq k$, and $x_i = 1$ implies $i \in \hat{N}$ for each $i \in N$. Let $\boldsymbol{X}$ be the set of all feasible allocations.

Now we are ready to give a formal description of (direct revelation) mechanisms[2]. A mechanism (f, t) consists of two components, an allocation rule f and a profile of transfer rules $(t_i)_{i \in N}$. An allocation rule f maps a profile $\boldsymbol{\theta}'$ of reported types to a feasible allocation $f(\boldsymbol{\theta}') \in \boldsymbol{X}$. We sometimes use the notation of $f(\theta'_i, \boldsymbol{\theta}'_{-i})$ instead, especially when we focus on the report of a specific buyer i, where $\boldsymbol{\theta}'_{-i}$ indicates the profile of types reported by the others. Given $\boldsymbol{\theta}'$, $f_i(\boldsymbol{\theta}') \in \{0, 1\}$ denotes the assignment to buyer i. Each transfer rule t_i maps a profile $\boldsymbol{\theta}'$ to a real number $t_i(\boldsymbol{\theta}') \in \mathbb{R}$, which indicates the amount that buyer i pays to the seller.

Here, we define several properties that mechanisms should satisfy. Feasibility requires that for any input, the allocation returned by the mechanism is feasible.

Definition 1. *A mechanism (f, t) is* feasible *if for any $\boldsymbol{\theta}'$, $f(\boldsymbol{\theta}')$ is feasible.*

Strategy-proofness is an incentive property, requiring that, for any buyer, reporting its true valuation and forwarding the information to all of its followers is a dominant strategy.

Definition 2. *Given a mechanism (f, t) and a buyer i with true type $\theta_i = (v_i, r_i)$, a report $\theta^*_i = (v^*_i, r^*_i) \in R(\theta_i)$ is a* dominant strategy *if for any $\boldsymbol{\theta}'_{-i}$ and $\theta'_i \in R(\theta_i)$,*

$$v_i \cdot f(\theta^*_i, \boldsymbol{\theta}'_{-i}) - t_i(\theta^*_i, \boldsymbol{\theta}'_{-i}) \geq v_i \cdot f(\theta'_i, \boldsymbol{\theta}'_{-i}) - t_i(\theta'_i, \boldsymbol{\theta}'_{-i})$$

holds. A mechanism (f, t) is strategy-proof *if reporting θ_i is a dominant strategy for any i under (f, t).*

Individual rationality is a property related to the incentives of the buyers for participation, which requires that truth-telling guarantees a non-negative utility.

Definition 3. *A mechanism (f, t) is* individually rational *if for any i, θ_i, and $\boldsymbol{\theta}'_{-i}$, $v_i \cdot f(\theta_i, \boldsymbol{\theta}'_{-i}) - t_i(\theta_i, \boldsymbol{\theta}'_{-i}) \geq 0$ holds.*

Non-deficit is a property about seller's revenue, which requires that the seller's revenue cannot be negative. Note that it does not consider each individual transfer and thus does not imply the non-negativity of each buyer's payment.

Definition 4. *A mechanism (f, t) satisfies* non-deficit *if for any $\boldsymbol{\theta}'$, $\sum_{i \in N} t_i(\boldsymbol{\theta}') \geq 0$ holds.*

Non-wastefulness is a property about the efficiency of allocation, which requires that the mechanism allocate as

[2]As discussed in the full version, the revelation principle holds under our assumption R on reportable subset $r'_i \subseteq r_i$. Therefore, focusing on direct revelation mechanisms is w.l.o.g.

many units as possible. Note that the traditional definition of non-wastefulness ignores the network structure, and thus the second term in RHS is replaced with $|N|$.

Definition 5. *A mechanism* (f, t) *is* non-wasteful *if for any* $\boldsymbol{\theta}'$, $\sum_{i \in N} f_i(\boldsymbol{\theta}') \geq \min\{k, |\hat{N}|\}$ *holds.*

2.1 Two Naïve Mechanisms for Comparison

One might expect that those properties hold in naïve mechanisms. Indeed, we can easily find the following two candidates. The formal definitions are in the full version. We compare their performances with that of our new mechanism in the following sections.

The first mechanism applies VCG to only the direct buyers r_s. It satisfies strategy-proofness, individual rationality, non-deficit, and non-wastefulness for $|r_s| \geq k$. We refer to this mechanism as *No-Diffusion-VCG* (ND-VCG in short). Such a mechanism is also considered in Li et al. (2017), although they focused on single-item auctions.

The second mechanism gives the units to buyers for free, in the first-come-first-served manner, which is referred to as *FCFS-F*. It satisfies individual rationality, non-deficit, and non-wastefulness, and it is strategy-proof when earlier arrivals are not allowed, e.g., based on ascending order of $d(\cdot)$, as usually assumed in online mechanism design (Hajiaghayi, Kleinberg, and Parkes 2004; Todo et al. 2012).

3 Distance-based Network Auction mechanism for Multi-unit, Unit-demand buyers

The definition of the new mechanism is given in Definition 6. A key concept in describing the mechanism is the *diffusion critical tree* $T(\boldsymbol{\theta}')$, originally introduced in Zhao et al. (2018). Given $\boldsymbol{\theta}'$, the diffusion critical tree $T(\boldsymbol{\theta}')$ is a rooted tree, where s is the root, the nodes of $T(\boldsymbol{\theta}')$ are all the connected buyers $\hat{N}$, and for each node $i \in \hat{N}$ and its least critical parent $j \in P(\boldsymbol{\theta}')$, an edge (j, i) is drawn. If $P_i(\boldsymbol{\theta}') = \emptyset$, we draw an edge (s, i). Furthermore, given $T(\boldsymbol{\theta}')$ and a node i, all of the nodes in the subtree of $T(\boldsymbol{\theta}')$ rooted at i are called i's *descendants*. Also, given a report $\boldsymbol{\theta}'$, a subset $S \subseteq \hat{N}$, and an integer $k' \leq k$, $v^*(S, k')$ denotes the k'-th highest value in S under $\boldsymbol{\theta}'$. For $k' \leq 0$, let $v^*(S, k') = \infty$. In addition, if $|S| < k'$, then $v^*(S, k') = 0$.

Definition 6. *Given* $\boldsymbol{\theta}'$*, first order the connected buyers* $\hat{N}$ *in ascending order of* $d(\cdot)$*, with arbitrary fixed tie-breaking. Note that* $d(\cdot)$ *is the distance from* s *in the original graph, not the distance in* $T(\boldsymbol{\theta}')$*. The order* $\succ$ *is called the* priority order. *For each* $i \notin \hat{N}$, $f_i(\boldsymbol{\theta}') = t_i(\boldsymbol{\theta}') = 0$. *For each* $i \in \hat{N}$*, let* $\hat{N}_{-i}$ *be the set of all connected buyers except* i *and its descendants in* $T(\boldsymbol{\theta}')$*. It then runs as follows:*

$k' \leftarrow k, W \leftarrow \emptyset$
for *each* $i \in \hat{N}$ *selected in the order of* $\succ$ **do**
  $p_i \leftarrow v^*(\hat{N}_{-i} \setminus W, k')$
  if $v'_i \geq p_i$ **then**
    $f_i(\boldsymbol{\theta}') \leftarrow 1, t_i(\boldsymbol{\theta}') \leftarrow p_i$
    $k' \leftarrow k' - 1, W \leftarrow W \cup \{i\}$
  else
    $f_i(\boldsymbol{\theta}') \leftarrow 0, t_i(\boldsymbol{\theta}') \leftarrow 0$
  end if
end for

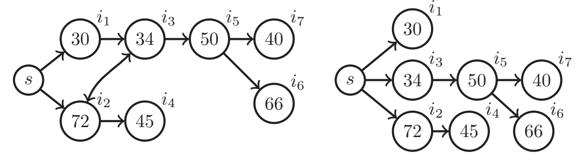

Figure 1: Example of Buyers Network and Corresponding Diffusion Critical Tree.

The following example demonstrates how the DNA-MU works. The network of buyers, defined by $\boldsymbol{r}$, as well as their true values, is shown in Fig. 1.

Example 1. *Consider three units and seven buyers* $N = \{i_1, i_2, \ldots, i_7\}$*. Each vertex in the left figure of Fig. 1 corresponds to a buyer, and the number in each vertex denotes her true valuation. The priority order is given as* $i_1 \succ i_2 \succ \cdots \succ i_7$*. Assume that every buyer forwards the information to all of her followers, i.e.,* $\hat{N} = N$*. The corresponding diffusion critical tree is given on the right in Fig. 1.*

The assignment to buyers is computed one-by-one, in the priority order. For buyer i_1*, the price is given as* $p_{i_1} = v^*(\hat{N}_{-i_1} \setminus W, k - |W|) = v^*(\{i_2, i_3, i_4, i_5, i_6, i_7\}, 3) = v_{i_5} = 50$*. Since* $p_{i_1} > v_{i_1}$*, she does not win a unit. For buyer* i_2*, the price is given as* $p_{i_2} = v^*(\{i_1, i_3, i_5, i_6, i_7\}, 3) = v_{i_7} = 40$*. Note that* v_{i_4} *is ignored because* i_2 *is a critical parent of* i_4*. Since* $v_{i_2} > p_{i_2}$*, she wins a unit;* W *is updated to* $\{i_2\}$*, and* k *is decremented to* 2*. For* i_3*, the price is given as* $p_{i_3} = v^*(\{i_1, i_4\}, 2) = v_{i_1} = 30$*. Since* $v_{i_3} > p_{i_3}$*, she wins a unit;* W *is updated to* $\{i_2, i_3\}$*, and* k *is decremented to* 1*. For* i_4*, the price is given as* $p_{i_4} = v^*(\{i_1, i_5, i_6, i_7\}, 1) = v_{i_6} = 66$*. Since* $p_{i_4} > v_{i_4}$*, she does not win a unit.*

For i_5*, the price is given as* $p_{i_5} = v^*(\{i_1, i_4\}, 1) = v_{i_4} = 45$*. Since* $v_{i_5} > p_{i_5}$*, she wins a unit;* W *is updated to* $\{i_2, i_3, i_5\}$ *and* k *is decremented to* 0*. Since no unit remains, the prices for the remaining buyers,* i_6 *and* i_7*, become infinity, and thus neither of the buyers wins a unit. To sum up,* i_2*,* i_3*, and* i_5 *are winners, who pay 40, 30, and 45 respectively.*

Let us clarify how it differs from GIDM by Zhao et al. (2018) and how it maintains strategy-proofness. GIDM first assigns, according to the reported $\boldsymbol{\theta}'$, a certain number of units to each subtree of $T(\boldsymbol{\theta}')$. The buyers in a subtree then compete with each other to buy the units assigned to it. This is something like creating a sub-market for each subtree. However, by not forwarding the information, some buyer, who originally loses due to the existence of some winning parent, can reduce the number of units assigned to the subtree, make the sub-market more competitive and the parent losing, and obtain a chance to win. This is actually the case found by Takanashi et al. (2019).

The DNA-MU also uses the diffusion critical tree. However, it does not create such a sub-market for each subtree. Instead, it has a single market with all of the units, where buyers' priorities are defined based on the distance $d(\cdot)$, which is not successfully manipulable; no buyer can make the distance shorter by not forwarding the information to her followers, which is shown by Lemma 1.

3.1 Properties of DNA-MU

We show feasibility, individual rationality, and non-deficit in Theorem 1, non-wastefulness in Theorem 2, and strategy-proofness in Theorem 3. The proofs of Theorems 1 and 2 are in the full version due to space limitations. Let $\hat{W}$ denote a set of winners $\{w_1, w_2, \dots\}$ and $\hat{W}_{\succ j}$ denote $\{w \in \hat{W} \mid w \succ j\}$.

Theorem 1. *The DNA-MU satisfies feasibility, individual rationality, and non-deficit.*

Theorem 2. *The DNA-MU is non-wasteful.*

Theorem 3. *The DNA-MU is strategy-proof.*

Proof. Let (f, t) be the DNA-MU. It suffices to show that (I) a buyer has no incentive not to forward information to her followers, and that (II) a buyer cannot obtain any gain by misreporting her value. That is, for any $\theta_i = (v_i, r_i)$ and $\theta'_i = (v'_i, r'_i) \in R(\theta_i)$, consider an intermediate type $\theta_i^m = (v_i, r'_i)$. The strategy-proofness condition thus holds from (I) $v_i \cdot f_i(\theta_i, \boldsymbol{\theta}'_{-i}) - t_i(\theta_i, \boldsymbol{\theta}'_{-i}) \geq v_i \cdot f_i(\theta_i^m, \boldsymbol{\theta}'_{-i}) - t_i(\theta_i^m, \boldsymbol{\theta}'_{-i})$ and (II) $v_i \cdot f_i(\theta_i^m, \boldsymbol{\theta}'_{-i}) - t_i(\theta_i^m, \boldsymbol{\theta}'_{-i}) \geq v_i \cdot f_i(\theta'_i, \boldsymbol{\theta}'_{-i}) - t_i(\theta'_i, \boldsymbol{\theta}'_{-i})$. These inequalities are proven in Lemmas 1 and 2. □

Lemma 1. *For any i, $\theta_i = (v_i, r_i)$, $\boldsymbol{\theta}'_{-i}$, and $\theta'_i = (v_i, r'_i)$ s.t. $r'_i \subset r_i$, $v_i \cdot f_i(\theta_i, \boldsymbol{\theta}'_{-i}) - t_i(\theta_i, \boldsymbol{\theta}'_{-i}) \geq v_i \cdot f_i(\theta'_i, \boldsymbol{\theta}'_{-i}) - t_i(\theta'_i, \boldsymbol{\theta}'_{-i})$ holds.*

Proof. By not forwarding the information, i can affect another buyer j in one of the following ways: (i) buyer j, who is originally a descendant of i in $T(\boldsymbol{\theta}')$, becomes disconnected, (ii) for buyer j, which originally satisfies $i \succ j$, the distance $d(j)$ becomes larger. In case (i), j is originally not included in $\hat{N}_{-i}$. Furthermore, making j disconnected might decrease the price of other buyers j' s.t. $j' \succ i$. Then there is a chance that i's price increases. Thus, not forwarding the information is useless in case (i). In case (ii), even when $d(j)$ becomes larger, $i \succ j$ holds originally, and i's price does not change. Thus, not forwarding the information is futile. □

Lemma 2. *For any i, $\theta_i = (v_i, r_i)$, $\boldsymbol{\theta}'_{-i}$, and $\theta'_i = (v'_i, r_i)$, $v_i \cdot f_i(\theta_i, \boldsymbol{\theta}'_{-i}) - t_i(\theta_i, \boldsymbol{\theta}'_{-i}) \geq v_i \cdot f_i(\theta'_i, \boldsymbol{\theta}'_{-i}) - t_i(\theta'_i, \boldsymbol{\theta}'_{-i})$ holds.*

Proof. For buyer i, her price p_i is given as: $v^*(\hat{N}_{-i} \setminus \hat{W}_{\succ i}, k - |\hat{W}_{\succ i}|)$. It is clear that $p_i \geq v^*(\hat{N}_{-i}, k)$ holds. Let π_i denote $v^*(\hat{N}_{-i}, k)$. π_i is determined independently from i's declared evaluation value. If $v_i \leq \pi_i$ holds, i cannot gain a positive utility regardless of her declaration. Thus, assume $v_i > \pi_i$ holds. Her actual price, i.e., $p_i = v^*(\hat{N}_{-i} \setminus \hat{W}_{\succ i}, k - |\hat{W}_{\succ i}|)$, can be strictly larger than π_i, if some buyer j (where $j \succ i$) s.t. $v'_j \leq \pi_i$ becomes a winner. Note that if $v'_j > \pi_i$ holds, j is within the top $k-1$ winners in $\hat{N}_{-i}$; the fact that j becomes a winner does not change p_i.

The only way for i to decrease her price is to turn such a winner into a loser by over-bidding. Assume j (where $j \succ i$) is such a winner. If j is i's ancestor, i cannot affect j's price. Thus, j and i are in different branches in $T(\boldsymbol{\theta}')$. Since j is a winner, $v'_j \geq v^*(\hat{N}_{-j} \setminus \hat{W}_{\succ j}, k - |\hat{W}_{\succ j}|)$ holds. Also, to increase j's price, v_i must be smaller than or equal to $v^*(\hat{N}_{-j} \setminus \hat{W}_{\succ j}, k - |\hat{W}_{\succ j}|)$. Note that i is included in $\hat{N}_{-j} \setminus \hat{W}_{\succ j}$. If i is within the top $k - |\hat{W}_{\succ j}| - 1$ buyers in $\hat{N}_{-j} \setminus \hat{W}_{\succ j}$, even if i over-bids, she cannot change j's price. Thus, $v'_j \geq v_i$ holds. However, we assume $v'_j \leq \pi_i < v_i$ holds. This is a contradiction. Thus, i cannot decrease her price by misreporting her evaluation value. □

4 Efficiency Analysis

In this section we conduct a more detailed analysis on efficiency. We show that any winner has a value that is in the set of top-k buyers except for her descendants. Also, the social surplus of the DNA-MU is always as large as those of the two naïve ones. Furthermore, the worst-case inefficiency of the DNA-MU can be bounded by choosing an appropriate reserve price.

4.1 Bounded Efficiency

Pareto efficiency in the multi-unit auction with k units requires that each buyer is a winner only if she is in the set of top-k buyers, i.e., whose value is more than or equal to the k-th highest value. However, it is not compatible with strategy-proofness in our model with the buyers' network, since a buyer would have an incentive for not forwarding information to her descendants if she needs to compete with them. Thus, we introduce a weaker concept called *bounded efficiency*, which is consistent with the incentive of buyers to forward the information. We say an allocation satisfies bounded efficiency if each winner is in the set of top-k buyers *except for its descendants*. Also, a mechanism satisfies bounded efficiency if it always obtains a bounded efficient allocation. By ignoring the descendants of each buyer, the incentive of information forwarding can still be guaranteed.[3] Indeed, our mechanism satisfies bounded efficiency.

Proposition 1. *The DNA-MU satisfies bounded efficiency: $\forall \boldsymbol{\theta}'$, $\forall i \in \hat{N}$ s.t. $f_i(\boldsymbol{\theta}') = 1$, $\#\{j \in \hat{N}_{-i} \mid v'_j > v'_i\} < k$.*

Proof. Let $i \in \hat{N}$ be an arbitrarily chosen winner and $W \subseteq \hat{N} \setminus \{i\}$ be the set of winners chosen before i in the mechanism. By definition, the winner i faces the price $v^*(\hat{N}_{-i} \setminus W, k - |W|)$. Since i is a winner, $v'_i \geq v^*(\hat{N}_{-i} \setminus W, k - |W|)$ holds, implying that there are less than $k - |W|$ buyers in $\hat{N}_{-i} \setminus W$, whose values are strictly larger than v'_i, i.e., $\#\{j \in \hat{N}_{-i} \setminus W \mid v'_j > v'_i\} < k - |W|$. Therefore, regardless of how many winners in W have a strictly larger value than v'_i, it holds that $\#\{j \in \hat{N}_{-i} \mid v'_j > v'_i\} < k$. □

[3]Note that the number of buyers, each of which is in the set of top-k buyers except its descendants, can be more than k. Thus, it is impossible to guarantee that all of them are winners.

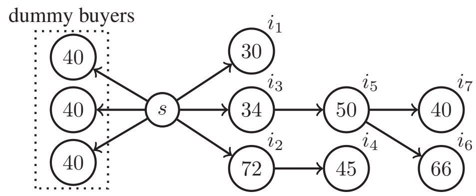

Figure 2: Implementation of Reserve Price $v_h(= 40)$ by Adding $k(= 3)$ Dummy Buyers to Diffusion Critical Tree.

This property is useful to show other characteristics of our mechanism, e.g., Proposition 2. One can also easily observe that the two naïve mechanisms violate this property.

4.2 Social Surplus Domination

A mechanism (f, p) is said to *dominate* another mechanism (f', p') *in terms of social surplus* if for any N and any $\boldsymbol{\theta}'$, it holds that $\sum_{i \in N} v_i \cdot f_i(\boldsymbol{\theta}') \geq \sum_{i \in N} v_i \cdot f'_i(\boldsymbol{\theta}')$.

Proposition 2. *The DNA-MU dominates both ND-VCG and FCFS-F in terms of social surplus, but not vice versa.*

Proof. When $|\hat{N}| \leq k$, every buyer receives a unit both in the distance-based mechanism and in ND-VCG. We then consider the cases of $|\hat{N}| > k$. First observe that, when $r_s = \hat{N}$, i.e., there only exist the direct buyers, the set of winners in both mechanisms coincides, so that the top-k buyers win a unit; this is obvious from the definition for ND-VCG, and it also holds for the DNA-MU from Proposition 1.

Furthermore, consider the following imaginary process. We start from the situation where only direct buyers exist, then we add other buyers one by one in the ascending order of their distances from the source. On one hand, a winner becomes a loser only when her value is lower than the value of a newly added buyer; the addition of a new buyer weakly increases the social surplus in DNA-MU. Also, there exists a case where the social surplus strictly increases. On the other hand, winners and the social surplus remain the same in ND-VCG. Thus, DNA-MU dominates ND-VCG but not vice versa. Using a similar argument, we can show that DNA-MU dominates FCFS-F, but not vice versa. $\square$

4.3 Worst-Case Efficiency Loss

When the seller wants to maximize revenue, it is natural to consider introducing a reserve price, i.e., the threshold bidding value for each buyer to own the right to win a unit (Myerson 1981). Letting v_h be the reserve price that the seller introduces, the DNA-MU with a reserve price v_h is then implemented by adding k dummy vertices with value v_h in $T(\boldsymbol{\theta}')$, each of which is connected only to s (see Fig. 2), while in line 2 of the algorithm the dummies are not considered. In other words, those dummies only affect $\hat{N}_{-i}$ for each $i \in \hat{N}$ and have no chance to win. The following example, which uses the same profile of the reports with Example 1, demonstrates how the introduction of a reserve price changes the allocation.

Example 2. *See Fig. 2. Since there are three units, the mechanism first adds three dummy vertices. The price for i_1 is given as $p_{i_1} = 50$, and she is not allocated a unit. The price for i_2 is given as $p_{i_2} = 45$, and she wins a unit. The price for i_3 is given as $p_{i_3} = 40$, which comes from the valuation of the dummy buyer. Since her value is strictly less than p_{i_3}, she is not allocated a unit. The price for i_4 is given as $p_{i_4} = 50$, and she is not allocated a unit. The price for i_5 is given as $p_{i_5} = 40$, and she wins a unit. At this moment one unit remains. For buyer i_6, the price is given as $p_{i_6} = 45$, which is strictly less than her value of* 66*. Thus, she wins a unit. Now that no unit remains, the price for buyer i_7 is set to be infinity, To sum up, i_2, i_5, and i_6 win a unit, and each pays* 40, 40, *and* 45, *respectively.*

Nearly identical proofs work for feasibility, non-deficit, individual rationality, and strategy-proofness. However, the introduction of a reserve price obviously breaks down non-wastefulness. Actually, for any non-zero v_h, there is a case where no buyer wins a unit, e.g., $v_i < v_h$ for every $i \in N$. This implies that, when we consider the approximation ratio an efficiency measure, the DNA-MU with reserve price performs poorly. Even worse, the original definition without a reserve price still has an arbitrarily worse (i.e., arbitrarily close to zero) approximation ratio.

Nevertheless, it remains important to clarify the effect of different reserve prices, given the practical usefulness of reserve prices. We therefore consider the following worst case efficiency measure called α-inefficiency, inspired by Nath and Sandholm (2018), and find that the optimal reserve price is $\bar{v}/2$, where $\bar{v}$ is the upper bound of the value, i.e., for each $i \in N$, $v_i \leq \bar{v}$.

Definition 7. *Let $\bar{v}$ be the upper bound of the value. A mechanism (f, t) is α-*inefficient *if*

$$\alpha = \frac{1}{k\bar{v}} \sup_{\boldsymbol{\theta}' \in \boldsymbol{\Theta}} \left[\max_{\boldsymbol{x} \in \boldsymbol{X}} \sum_{i \in N} v'_i \cdot x_i - \sum_{i \in N} v'_i \cdot f_i(\boldsymbol{\theta}') \right].$$

The range of α is $[0, 1]$, and having a smaller α is better. We first provide a lemma that is useful to provide the worst-case inefficiency, while its proof appears in the full version due to space limitations. Given $\boldsymbol{\theta}'$, let ℓ denote the number of connected buyers whose values are no less than v_h, i.e., $\ell := \#\{i \in \hat{N} \mid v_i \geq v_h\}$.

Lemma 3. *Assuming all buyers declare their true values,* $\min(\ell, k)$ *units are allocated in the DNA-MU with a reserve price.*

Given the above lemma, we show that the DNA-MU with $v_h = \bar{v}/2$ satisfies $1/2$-inefficiency.

Theorem 4. *The DNA-MU with reserve price v_h satisfies $1/2$-inefficiency by setting $v_h = \bar{v}/2$.*

Proof. If $\ell < k$, the DNA-MU allocates units to the top ℓ buyers within $\hat{N}$ (in terms of values) from Lemma 3. The remaining $k - \ell$ units cannot be allocated since the values of other buyers are less than v_h. Thus, the maximum efficiency loss is bounded by $(k - \ell)v_h$ (if $k - \ell$ buyers exist whose values are $v_h - \epsilon$, the efficiency loss becomes $(k - \ell)(v_h - \epsilon)$).

In particular, if the value of each buyer is less than v_h, ℓ becomes 0. Thus, the worst case efficiency loss is $k \cdot v_h$. If $\ell \geq k$, the DNA-MU allocates k units from Lemma 3. The maximum efficiency loss is bounded by $k(\bar{v}-v_h)$, which can occur, for instance, when there are $2k$ buyers, forming a path graph, and those k buyers closer to s have the value of v_h, while the rest have the value of $\bar{v}$: the DNA-MU allocates k units to the closest k buyers. From the above, the maximum efficiency loss is given as $\max(k \cdot v_h, k(\bar{v} - v_h))$. This is bounded from the bottom by $k \cdot \bar{v}/2$, which is achieved by setting v_h to $\bar{v}/2$. Thus, the DNA-MU is $1/2$-inefficient for $v_h = \bar{v}/2$. □

Observe that there is a tradeoff between achieving non-wastefulness and guaranteeing a better worst-case performance by the mechanism with a reserve price, where the former is achieved by $v_h = 0$ and the latter by $v_h = \bar{v}/2$. Obtaining the lower bound of α that a strategy-proof mechanism achieves remains an open question. However, since 0-inefficiency implies Pareto efficiency, the impossibility suggested by Takanashi et al. (2019) implies that no strategy-proof mechanism that also satisfies non-deficit and individual rationality achieves 0-inefficiency.

5 Revenue Analysis

The seller's revenue is also an important evaluation criterion for auction mechanisms. In this section, we first show that the seller's revenue in the DNA-MU is no less than those of the two naïve ones. We also show that maximizing the revenue by optimally choosing the set of its followers to whom it sends the information is NP-complete.

5.1 Revenue Domination

We define the domination in terms of the seller's revenue analogously. A mechanism (f, p) *dominates* another mechanism (f', p') *in terms of the seller's revenue* if for any N and any $\boldsymbol{\theta}'$, it holds that $\sum_{i \in N} t_i(\boldsymbol{\theta}') \geq \sum_{i \in N} t'_i(\boldsymbol{\theta}')$.

Proposition 3. *The DNA-MU dominates both ND-VCG and FCFS-F in terms of the seller's revenue, but not vice versa.*

Proof. The DNA-MU obviously dominates ND-VCG when $|r_s| \leq k$, since the price for each winner in ND-VCG is zero. When $|r_s| > k$, each winner in ND-VCG pays $v^*(r_s \setminus \{i\}, k)$. On the other hand, the price p_i for each winner i in the DNA-MU satisfies $p_i \geq v^*(\hat{N}_{-i}, k)$ by definition. For every winner i, $\hat{N}_{-i}$ is a superset of $r_s \setminus \{i\}$. Therefore, from the monotonicity of v^* on the first argument, $p_i \geq v^*(\hat{N}_{-i}, k) \geq v^*(r_s \setminus \{i\}, k)$ holds. Also, there exists a case where the inequality becomes strict. Thus, the DNA-MU dominates ND-VCG but not vice versa. Since the revenue of FCFS-F is always zero, while the revenue of the DNA-MU is non-negative for any input and can be strictly positive, the DNA-MU also dominates FCFS-F but not vice versa. □

5.2 Revenue Monotonicity

The seller's revenue is required to have some specific form of monotonicity. Several forms of such *revenue monotonicity* have been studied, including bidder revenue monotonicity (Rastegari, Condon, and Leyton-Brown 2011; Todo, Iwasaki, and Yokoo 2010) and item revenue monotonicity, a.k.a. destruction-proofness (Muto and Shirata 2017).

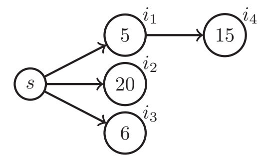

Figure 3: Violation of Revenue Monotonicity.

The condition studied in this section is weaker than bidder revenue monotonicity. A mechanism is *follower revenue monotonic* if the seller's revenue is monotonically increasing with respect to the number of direct buyers. Let $t_i(\boldsymbol{\theta}' \mid r'_s)$ be the payment from buyer i when $\boldsymbol{\theta}'$ is reported and s sends the information of the auction to a subset r'_s of direct buyers.

Definition 8. *A mechanism (f, t) is follower revenue monotonic if for any r_s, $\boldsymbol{\theta}'$, and $r'_s \subseteq r_s$, it holds that $\sum_{i \in N} t_i(\boldsymbol{\theta}' \mid r_s) \geq \sum_{i \in N} t_i(\boldsymbol{\theta}' \mid r'_s)$.*

For a multi-unit auction without any network among buyers, there are several bidder revenue monotonic mechanisms, and thus follower revenue monotonic ones. However, the example below shows that, for auction via social network, our mechanism is not even follower revenue monotonic.

Example 3. *Consider two units and four buyers i_1, i_2, i_3, and i_4; see Fig. 3. The priority order is $i_1 \succ i_2 \succ i_3 \succ i_4$.*

Assume all of the buyers behave sincerely. When the seller s sends the information to all of its followers, i.e., i_1, i_2, and i_3, all buyers become connected. The price for each buyer is given as $p_1 = 6$, $p_2 = 6$, $p_3 = 15$, and $p_4 = 6$, where i_2 and i_4 win a unit and the revenue is 12. When s sends the information only to i_1 and i_2, i_3 becomes unconnected. The price for each buyer is given as $p_1 = 0$, $p_2 = 15$, and $p_4 = \infty$, where i_1 and i_2 win a unit and the revenue is 15. As a result, the revenue is not maximized when the seller sends the information to all the direct buyers.

Therefore, a question rises to the seller: to which set of direct buyers should she send the information to maximize her revenue? We define a simplified form of this problem, so called OPTIMAL DIFFUSION, from the perspective of computational complexity: Assuming that the seller knows the exact network among buyers and that all buyers behave sincerely, she tries to find an optimal set of direct buyers to whom she should send the information.

Definition 9 (OPTIMAL DIFFUSION)**.** *Given a number k of units, a profile $\boldsymbol{\theta}'$, a set r_s of direct buyers, and a threshold K, is there a subset $r'_s \subseteq r_s$ s.t. $\sum_{i \in N} t_i(\boldsymbol{\theta}' \mid r'_s) \geq K$ holds under the DNA-MU?*

We show that OPTIMAL DIFFUSION is NP-complete by a reduction from PARTITION. Due to space limitations, we present a proof sketch below; a full proof is in the full version.

Definition 10 (PARTITION)**.** *Given a set $\boldsymbol{A}$ where each $i \in$*

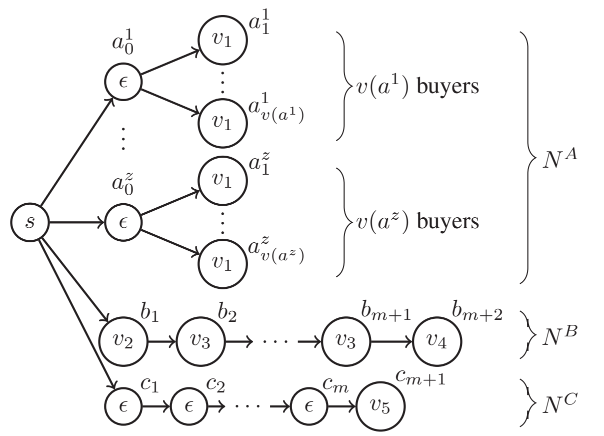

Figure 4: Reduction from Partition: Network of Buyers.

$\boldsymbol{A}$ has a value $v(i) \in \mathcal{Z}^+$, does there exist a subset $\boldsymbol{A'} \subseteq \boldsymbol{A}$ such that $\sum_{i \in \boldsymbol{A'}} v(i) = m$, where $m = \sum_{i \in \boldsymbol{A}} v(i)/2$?

Theorem 5. OPTIMAL DIFFUSION *is NP-complete.*

Proof Sketch. First, OPTIMAL DIFFUSION is in NP since we can compute $\sum_{i \in N} t_i(\boldsymbol{\theta}' \mid r'_s)$ in polynomial time. Given an instance of PARTITION, we construct an instance of OPTIMAL DIFFUSION as follows, with $N = N^A \cup N^B \cup N^C$:

- For all $i \in \boldsymbol{A}$, we create set $(a^i_j)_{0 \le j \le v(i)}$ in N^A such that $\theta_{a^i_0} = (\epsilon, (a^i_j)_{1 \le j \le v(i)})$, and $\theta_{a^i_j} = (v_1, \emptyset)$ for $1 \le j \le v(i)$.
- Set $N^B = (b_j)_{1 \le j \le m+2}$ is such that $\theta_{b_1} = (v_2, \{b_2\})$, $\theta_{b_j} = (v_3, \{b_{j+1}\})$ for $2 \le j \le m+1$, and $\theta_{b_{m+2}} = (v_4, \emptyset)$.
- Set $N^C = (c_j)_{1 \le j \le m+1}$ is such that $\theta_{c_j} = (\epsilon, \{c_{j+1}\})$ for $1 \le j \le m$, and $\theta_{c_{m+1}} = (v_5, \emptyset)$.
- The seller's direct followers are $(a^i_0)_{i \in A} \cup \{b_1, c_1\}$.

The network is illustrated in Fig. 4. Buyers are labelled with any ascending order of $d(\cdot)$ satisfying $b_{m+1} \succ c_{m+1}$. The prices satisfy $\epsilon < v_2 < v_1 < v_3 << v_4 < v_5$. The number of units is $k = m + 2$ and the threshold is $K = \epsilon + m \cdot v_1 + v_4$.

We briefly argue the validity of the reduction. Notice that buyers b_1 and c_1 belong to any $r'_s \subseteq r_s$ such that $\sum_{i \in N} t_i(\boldsymbol{\theta} \mid r'_s) \ge K$, since otherwise price v_4 cannot be reached.

If $|\{a \in r'_s \mid v_a = v_1\}| = m$ holds, i.e., exactly m descendants with value v_1 can be chosen (thus the original PARTITION is "yes"), then buyer b_1 buys at price ϵ, buyers $(b_i)_{2 \le i \le m+1}$ at price v_1, and buyer c_{m+1} at price v_4. Hence, $\sum_{i \in N} t_i(\boldsymbol{\theta} \mid r'_s) = \epsilon + m \cdot v_1 + v_4 = K$ and OPTIMAL DIFFUSION is "yes".

If the original PARTITION is "no", either (i) $|a \in r'_s \mid v_a = v_1| < m$ or (ii) $|a \in r'_s \mid v_a = v_1| > m$ holds. In the case (i), buyers b_1 and b_2 buy at price ϵ. Hence, $\sum_{i \in N} t_i(\boldsymbol{\theta} \mid r'_s) < K$ and OPTIMAL DIFFUSION is "no".

In the case (ii), buyer b_1 does not buy, and c_{m+1} buys at price lower than v_4. Hence, $\sum_{i \in N} t_i(\boldsymbol{\theta} \mid r'_s) < K$ and OPTIMAL DIFFUSION is "no". □

6 Incentive Analysis

Now we show that, compared with those two naïve mechanisms, our mechanism also has its own strength on buyers incentive; in those mechanisms, hiding the information, combined with the report of the true value, is also a dominant strategy, while this is not the case in our mechanism for any $k \ge 2$. This indicates that the incentive for each buyer to report her type truthfully in the DNA-MU is stronger than that in both of those naïve ones.

Proposition 4. *Assume $k \ge 2$. For each i, reporting $(v_i, \emptyset)$ is not a dominant strategy in the DNA-MU.*

Proof. Consider k units and $k+2$ buyers $i_1, \dots, i_{k+2}$, such that $r_s = \{i_1, i_3, i_5, i_6, \dots, i_{k+2}\}$, $\theta_{i_1} = (15, \{i_2\})$, $\theta_{i_2} = (20, \emptyset)$, $\theta_{i_3} = (10, \{i_4\})$, $\theta_{i_4} = (9, \emptyset)$, and $\theta_{i_j} = (30, \emptyset)$ for all $5 \le j \le k+2$. The priority is given as $i_5 \succ i_6 \succ \dots \succ i_{k+2} \succ i_3 \succ i_1 \succ i_4 \succ i_2$. The first $k-2$ units are sold to $\{i_j\}_{5 \le j \le k+2}$, regardless of i_1's forwarding strategy. Under i_1's sincere forwarding to i_2, i_1 wins a unit and pays 9. If i_1 does not forward the information to i_2, then i_1 would win a unit and pay 10. So not forwarding the information is dominated by a sincere forwarding in this case. □

Proposition 5. *For each i, reporting $(v_i, \emptyset)$ is a dominant strategy in both ND-VCG and FCFS-F.*

Proof. In ND-VCG, only the reports from the direct buyers affect the outcome. Therefore, for each direct buyer $i \in r_s$, any valuation report v'_i, and any action by other buyers, the choice of $r'_i \subseteq r_i$ does not change the outcome at all.

In FCFS-F, for each $i \in N$, any follower of i who originally arrives after i under i's sincere forwarding r_i is still arriving after i under any manipulation $r'_i \subset r_i$. Thus, whether i wins a unit does not depend on the choice of r'_i. □

7 Conclusions

The DNA-MU satisfies strategy-proofness, non-wastefulness, non-deficit, and individual rationality. The performance is comprehensively analyzed; it dominates the two naïve mechanisms in terms of both social surplus and revenue. Several other properties are also revealed.

A more detailed analysis on the complexity of maximizing the seller's revenue is required, such as for the case with a fixed number k of units. Our future work will also include more general revenue analysis, e.g., revenue equivalence (Heydenreich et al. 2009) and revenue optimality (Myerson 1981). Extending the DNA-MU for more general domains, such as multi-unit auctions with decreasing marginal values, is also crucial. Considering an obviously strategy-proof auction via social network will also be an interesting direction (Li 2017).

8 Acknowledgments

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References

Borgatti, S. P.; Mehra, A.; Brass, D. J.; and Labianca, G. 2009. Network analysis in the social sciences. *Science* 323(5916):892–895.

Clarke, E. H. 1971. Multipart pricing of public goods. *Public Choice* 2:19–33.

Emek, Y.; Karidi, R.; Tennenholtz, M.; and Zohar, A. 2011. Mechanisms for multi-level marketing. In *Proceedings of the 12th ACM Conference on Electronic Commerce*, EC '11, 209–218. New York, NY, USA: ACM.

Groves, T. 1973. Incentives in teams. *Econometrica* 41:617–631. Microeconomic Theory,.

Hajiaghayi, M. T.; Kleinberg, R. D.; and Parkes, D. C. 2004. Adaptive limited-supply online auctions. In *the Fifth ACM Conference on Electronic Commerce (EC)*, 71–80.

Heydenreich, B.; Müller, R.; Uetz, M.; and Vohra, R. V. 2009. Characterization of revenue equivalence. *Econometrica* 77(1):307–316.

Jackson, M. O. 2008. *Social and Economic Networks*. Princeton University Press.

Kempe, D.; Kleinberg, J.; and Tardos, E. 2003. Maximizing the spread of influence through a social network. In *Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '03, 137–146. New York, NY, USA: ACM.

Li, B.; Hao, D.; Zhao, D.; and Zhou, T. 2017. Mechanism design in social networks. In *Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence, February 4-9, 2017, San Francisco, California, USA.*, 586–592.

Li, S. 2017. Obviously strategy-proof mechanisms. *American Economic Review* 107(11):3257–87.

Muto, N., and Shirata, Y. 2017. Manipulation via endowments in auctions with multiple goods. *Mathematical Social Sciences* 87:75 – 84.

Myerson, R. B. 1981. Optimal auction design. *Mathematics of Operations Research* 6(1):58–73.

Nath, S., and Sandholm, T. 2018. Efficiency and budget balance in general quasi-linear domains. *Games and Economic Behavior* 113:673–693.

Rastegari, B.; Condon, A.; and Leyton-Brown, K. 2011. Revenue monotonicity in deterministic, dominant-strategy combinatorial auctions. *Artificial Intelligence* 175(2):441–456.

Takanashi, S.; Kawasaki, T.; Todo, T.; and Yokoo, M. 2019. Efficiency in truthful auctions via a social network.

Todo, T.; Mouri, T.; Iwasaki, A.; and Yokoo, M. 2012. False-name-proofness in online mechanisms. In *Proceedings of the Eleventh International Conference on Autonomous Agents and Multiagent Systems*, to appear.

Todo, T.; Iwasaki, A.; and Yokoo, M. 2010. Characterization of revenue monotonicity in combinatorial auctions. In *Proceedings of the 2010 IEEE/WIC/ACM International Conference on Intelligent Agent Technology (IAT)*, 383–390.

Vickrey, W. 1961. Counter speculation, auctions, and competitive sealed tenders. *Journal of Finance* 16:8–37.

Zhao, D.; Li, B.; Xu, J.; Hao, D.; and Jennings, N. R. 2018. Selling multiple items via social networks. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS 2018, Stockholm, Sweden, July 10-15, 2018*, 68–76.